Name: ______________________________________
ID #: ______________________________________
Section #: ___________________________ Serial Number: ________________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.

2. Show all your work. No points for answers without justification.

3. Calculators and Mobiles are not allowed.

4. Make sure that you have 8 different problems (8 pages + cover page)

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1. If \( y_1 = x \cos(\ln x) \) is a solution of the differential equation:

\[
x^2 y'' - xy' + 2y = 0,
\]

use only the reduction of order to find a second solution \( y_2 \).

**Standard Form:** \( y'' - \frac{1}{x} y' + \frac{2}{x^2} y = 0 \)

\[ P(x) = -\frac{1}{x} \quad f(x) = 0 \]

\[
\int P(x) \, dx = \int \frac{dx}{x} = \ln x \quad 01
\]

\[
y_1(x) = y_1(x) \int -\frac{\int P(x) \, dx}{y_1^2} \, dx = x \cos(\ln x) \int \frac{x}{x^2 \cos^2(\ln x)} \, dx \quad 02
\]

Let \( t = \ln x \), so

\[
\int \frac{dx}{x} \cos(\ln x) = \int \frac{dt}{\cos^2 t} = \tan t = \tan(\ln x) \quad 02
\]

Hence \( y_2(x) = x \cos(\ln x) \cdot \tan(\ln x) \quad 01 \)

\[
y_2(x) = x \sin(\ln x)
\]
2. Given that \( y_1(x) = -\cos(x) \ln(\sec x + \tan x) \) is a solution of \( y'' + y = \tan x \),

(a) Find a particular solution of the differential equation:
\[ y'' + y = 5x + 8. \]

1st method: Clearly \( y_p = 5x + 8 \) is a particular solution of \( y'' + y = 5x + 8 \).

2nd method: Let \( y_p = Ax + B \) be a particular solution of \( y'' + y = 5x + 8 \).
So \( y' = A \) and \( y'' = 0 \) so \( Ax + B = 5x + B = 0 \) gives \( A = 5 \) and \( B = 8 \). Hence \( y_p = 5x + 8 \) is a particular solution of \( y'' + y = 5x + 8 \).

(b) Find a particular solution of the differential equation:
\[ y'' + y = 2\tan x + 5x + 8. \]

1st method: \( y_p = -2\cos(x)\ln(\sec x + \tan x) \) is a particular solution of \( y'' + y = 2\tan x \).

2nd method: From question (a), \( y_p = 5x + 8 \) is a particular solution of \( y'' + y = 5x + 8 \). Hence by the Superposition principle, \( y_p = y_{p1} + y_{p2} \) is a particular solution of:
\[ y'' + y = 2\tan x + 5x + 8. \]
3. Find the solution to the initial value problem:

\[
\begin{align*}
\begin{cases}
y'''' + y''' + y'' + y' + y &= 0 \\ y(0) &= 2, \ y'(0) = y''(0) = 0
\end{cases}
\end{align*}
\]

- Aux. eq: \( m^3 + m^2 + m + 1 = 0 \) = \((m+1)(m^2+1)\) \hspace{1cm} (02)
- \( m_1 = -1 \) is a root \hspace{1cm} (01)
- \( m_2 = i \), \( m_3 = -i \) / \( m_2 = \alpha + i \beta \) with \( \alpha = 0 \) and \( \beta = 1 \)

The General Solution is then given by:

\[
y = C_1 e^x + C_2 \cos x + C_3 \sin x
\]

\( y' = -C_1 e^x - C_2 \sin x + C_3 \cos x \) \hspace{1cm} (01)

\( y'' = -C_1 e^x - C_2 \cos x - C_3 \sin x \) \hspace{1cm} (01)

\[
\begin{align*}
\begin{cases}
y(0) &= 2 \Rightarrow C_1 + C_2 = 2 \quad \Rightarrow 2C_1 = 2 \\
y'(0) &= 0 \Rightarrow -C_1 + C_3 = 0 \quad \Rightarrow C_1 = C_3 \\
y''(0) &= 0 \Rightarrow C_1 - C_2 = 0 \quad \Rightarrow C_1 = C_2
\end{cases}
\end{align*}
\]

So \( C_1 = C_2 = C_3 = 1 \)

Hence The General Solution is given by:

\[
y = e^x + \cos x + \sin x
\]
4. Find the annihilator of the function

\[ 8e^{3x} \sin x + xe^{-\frac{1}{3}x} + x - 2. \]

\[ \text{x) For } 8e^{3x} \sin x \ (\alpha = 3, \beta = 7 \text{ and } n = 7) \]

So \( (D^2 - 6D + 10)(8e^{3x} \sin x) = 0 \)

\[ \text{x) For } xe^{-\frac{1}{3}x} \ (n = 2, \alpha = -\frac{1}{3}) \]

So \( (D + \frac{1}{3})^2 (xe^{-\frac{1}{3}x}) = 0 \)

\[ \text{x) For } x-2 : \ D^2(x-2) = 0 \]

Hence the annihilator of the function

\[ 8e^{3x} \sin x + xe^{-\frac{1}{3}x} + x - 2 \]

is

\[ D^2(D + \frac{1}{3})^2(D^2 - 6D + 10) \]
5. Find the general solution of the differential equation:

\[ y'' + 2y' = 4xe^{-2x} + 2\cos x \]

by the **undetermined coefficients** method. (Do not calculate the constant coefficients of the particular solution).

**Step I: Find \( y_c \)?

Homogeneous equation:

\[ y'' + 2y' = 0 \]

Auxiliary equation:

\[ m^2 + 2m = 0 \]

\[ m(m+2) = 0 \]

\[ m_1 = 0 \quad \text{and} \quad m_2 = -2 \]

So, \( y_c = c_1 + c_2 e^{-2x} \)

**Step II: Find \( y_p \)?

Since \( (D+2)^2(4xe^{-2x}) = 0 \) and \( (D^2+1)(2\cos x) = 0 \)

Then \( (D^2+1)(D+2)^2(D^2+2D)y = 0 \)

\[ D(D+2)^3(D^2+1)y = 0 \]

Auxiliary equation:

\[ m(m+2)^3(m^2+1) = 0 \]

\[ m_1 = 0 \]

\[ m_2 = -2 \quad (3 \text{ thms}) \]

\[ m_3 = i, \quad m_4 = -i \]

The General Solution is then given by:

\[ y = c_1 + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 x^2 e^{-2x} + c_5 \cos x + c_6 \sin x \]

Hence, the Basic Form of the particular solution is:

\[ y_p = Ax e^{2x} + Bx^2 e^{2x} + C \cos x + D \sin x \]

and the General Solution will be:

\[ y = c_1 + c_2 e^{-2x} + Ax e^{2x} + Bx^2 e^{2x} + C \cos x + D \sin x \]
6. Find the general solution of the differential equation:

\[ y'' + 8y' + 16y = \frac{1}{x^2} e^{-4x}, \quad x > 0 \]

by using the method of **variation of parameter**.

**Step I:**

**Hom. DE:** \[ y'' + 8y' + 16y = 0; f(x) = \frac{1}{x^2} e^{-4x} \]

**Aux. equation:** \[ m^2 + 8m + 16 = 0 = (m+4)^2 \]

\[ \Rightarrow \quad y_c = C_1 e^{-4x} + C_2 xe^{-4x} \]

**Step II:**

Let \( y_1 = e^{-4x} \) and \( y_2 = xe^{-4x} \)

Find \( y_p \) such that:

\[ y_p = u_1 y_1 + u_2 y_2 \]

\[ w(y_1, y_2) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4xe^{-4x} & -e^{-4x} \end{vmatrix} = -8x - 8x - 8xe^{-4x} = -8xe^{-4x} \]

\[ u_1' = -\frac{y_2 f(x)}{w} = -\frac{xe^{-4x} \cdot \frac{1}{x^2} e^{-4x}}{-8xe^{-4x}} = -\frac{1}{x} \Rightarrow u_1 = -\ln x \]

\[ u_2' = \frac{y_1 f(x)}{w} = \frac{e^{-4x} \cdot \frac{1}{x^2} e^{-4x}}{-8x} = \frac{1}{x^2} \Rightarrow u_2 = -\frac{1}{x} \]

So \( y_p = u_1 y_1 + u_2 y_2 = -e^{-4x} \ln x - xe^{-4x} = -xe^{-4x}(\ln x + 1) \)

Hence, the general solution is:

\[ y = y_c + y_p = C_1 e^{-4x} + C_2 xe^{-4x} - xe^{-4x}(\ln x + 1) \]
7. (i) Find the general solution of the differential equation

\[ 4t^2 y'' + 8ty' + y = 0 \]  

(1)

directly as a **Cauchy-Euler equation**.

Let \( y = t^m \) \( \Rightarrow y' = mt^{m-1} \), \( y'' = m(m-1)t^{m-2} \)

Substitute in (1):

\[ 4t^2 m(m-1)t^{m-2} + 8tm^{m-1} + t^m = 0 = t^m[4m(m-1) + 8m + 1] \]

Aux. eq: \( 4m(m-1) + 8m + 1 = 0 \) or \( 4m^2 + 4m + 1 = 0 \)

\[ 4m^2 + 4m + 1 = 0 = m^2 + m + \frac{1}{4} = \left(m + \frac{1}{2}\right)^2 \]

\( m_1 = m_2 = -\frac{1}{2} \)

Hence the **General Solution** is given by:

\[ y = c_1 t^{-\frac{1}{2}} + c_2 t^{-\frac{1}{2}} \ln t \]

(ii) Transform the differential equation (1) into a differential equation with constant coefficients (**Do not solve the obtained differential equation**).

Let \( t = e^x \) (or \( x = \ln t \)), we have: \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{t} \frac{dy}{dx} \)

\[ \frac{d^2y}{dt^2} = \frac{1}{t^2} \left( \frac{1}{t} \frac{dy}{dx} \right) = \frac{1}{t^2} \left( \frac{d^2y}{dx^2} - \frac{dy}{dx} \right) \]

Substitute in equation (1):

\[ 4t^2 \left( \frac{1}{t^2} \left( \frac{d^2y}{dx^2} - \frac{dy}{dx} \right) \right) + 8t \left( \frac{1}{t} \frac{dy}{dx} \right) + y = 0 = 4 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y \]

which is a **DE with Constant Coefficients**.
8. Find the Cauchy-Euler equation whose solution is given by:

\[ y = c_1 x + c_2 x \cos(\ln x) + c_3 x \sin(\ln x). \]

From the solution, the roots of the auxiliary equation are:

\[ m_1 = 1, \quad m_2 = 1 + i \quad \text{and} \quad m_3 = 1 - i. \]

Then:

Auxiliouar\_eq: \( (m-1)(m^2 - 2m + 2) = 0 = (m-1)(m-2) + 2m - 2 \)

Then \( m^3 - 3m^2 + 4m - 2 = 0 = m(m-1)(m-2) + 2m - 2 \)

Hence the Cauchy-Euler equation is:

\[ x^3 y''' + 2x y' - 2y = 0. \]