Exercise 1 (15 points)
Use the augmented matrix to find all values of \( r \) for which the system (S) has:

- a/ No solution
- b/ a unique solution
- c/ infinitely many solutions

(S) \[
\begin{align*}
\begin{bmatrix}
x + y - z &= 3 \\
-x + y + rz &= r \\
x - y - z &= 2
\end{bmatrix}
\end{align*}
\]
Exercise 2 (20 points)

Let \( N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \) and \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \)

1. Find \( N^2 \), \( N^3 \) and \( N^n \) for all \( n \geq 3 \).
2. Find \( A^n \) for all \( n \geq 3 \). [Hint: remark that \( A = I + N \)]
3. Use question 2- to find \( A^{100} \)
Exercise 3 (15 points)
Use elementary row operations to find the inverse of the matrix

\[
A = \begin{pmatrix}
1 & 1 & -1 \\
1 & 1 & 1 \\
2 & -1 & 1
\end{pmatrix}
\]
Exercise 4 (20 points)
Find all 3x3 matrices $A$ such that $AB = BA$ for every 3x3 matrix $B$
Exercise 5 (15 points)
Determine whether $W$ is a subspace of the given space $V$.

1. $V = K[X]$, the vector space of polynomials with coefficients in $K$, and $W = \{ f \in K[X] \mid f(3) = 0 \}$

2. $V = \mathbb{R}_3$ and $W = \{ (a, b, c) \in \mathbb{R}_3 \mid 2a - b + 2c = 0 \}$

3. $V = \mathbb{R}_3$ and $W = \{ (a, b, c) \in \mathbb{R}_3 \mid a - b + e^c = 1 \}$
Exercise 6 (15 points)
Let $A$ be an nxn symmetric matrix and $B$ an nxn skew symmetric matrix.
1- Find all values of $\alpha$ such that $A + (\alpha^2 - 1)B$ is symmetric.

2- Under which condition $AB$ is symmetric?

3- Find a 2x2 symmetric matrix $A$ and a 2x2 skew symmetric matrix $B$ such that $AB = -BA$