Exercise 1 (20 points)
Use the augmented matrix to find all values of \( r \) for which the system (S) has:

\[ \begin{align*}
    x + 2y + z &= 4 \\
    x - 2y + r^2z &= r \\
    x + 6y + z &= 7 \\
\end{align*} \]

a/ No solution       b/ a unique solution       c/ infinitely many solutions
Exercise 2 (20 points)
1-Which one of the following transformation is linear?
(i) $L_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3, L_1(a,b,c) = (a,b,c+1)$,
(ii) $L_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3, L_2(a,b,c) = (a-b, b-c, c-a)$
2-Find $\text{Ker}L_2$, and $\dim \text{Ker}L_2$
3-Find $\text{Range}L_2$ and $\dim \text{Range}L_2$
Exercise 3 (20 points)
Let $P_3$ be the vector space of all real polynomials of degree $\leq 3$, $P_2$ be the vector space of all real polynomials of degree $\leq 2$ and $D : P_3 \rightarrow P_2, D(f) = f'$ be the differential operator. Let $S = \{1, t, t^2, t^3\}$ and $T = \{1, t, t^2\}$ be the standard bases of $P_3$ and $P_2$ respectively. Set $S' = \{2, 1 - t, -t^2, t^2 - t^3\}$ and $T' = \{1, 1 - t, 1 - t^2\}$.

1- Find the transition matrix $P$ from $S'$ to $S$.
2- Find the transition matrix $Q$ from $T'$ to $T$.
3- Find the matrix representing $D$ with respect to $S'$ and $T'$.
Exercise 4 (20 points)

Use Gram-Schmidt process to transform the basis \( S = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) to an orthonormal basis.
Exercise 5 (20 points)
Let $V$ and $W$ be two vector spaces over the same field $k$ and suppose that $\dim V = \dim W$ is finite.
1-Prove that $V$ and $W$ are isomorphic.
2-Deduce that every real vector space $V$ of dimension $n$ is isomorphic to $\mathbb{R}^n$. 
Exercise 6 (20 points)
1-Use the properties of determinant to set-up a formula for the area of the rectangle ($\Delta$) with vertices $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$.
2-Application: Find the area of the triangle with vertices (1,2), (2,4) and (3,1).
**Exercise 7** (20 points)

Let $A$ be an $n \times n$ matrix with characteristic polynomial $f = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ and suppose that $a_0 \neq 0$.

1- Prove that $A$ is invertible and find $A^{-1}$. (Hint, use Cayley-Hamilton Theorem).

2- Application: Find the $3 \times 3$ matrix $A$ such that $f = -X^3 + 2X^2 - 1$ and $2A - A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
Exercise 8 (20 points)

Let \( A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \).

1. Prove that \( A \) is diagonalizable.

2. Find an orthogonal matrix \( P \) such that \( P^{-1}AP = D \) is diagonal.
Exercise 9 (20 points)

Let $A$ be a real symmetric matrix.

1-Prove that the eigenvalues of $A$ are all real numbers.
2-Prove that eigenvectors associated to distinct eigenvalues are orthogonal.
Exercise 10 (20 points)
Let \( g(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz + 2yz \) be a quadratic form of \( \mathbb{R}^3 \).
1- Find the canonical quadratic form \( h \) that is equivalent to \( g \).
2- Find the rank and the signature of \( g \).