Exercise 1. Let $U = (1, 1, 0)$ and $V = (1, 1, \sqrt{2})$.

(1) Evaluate the angle between the two vectors $U$ and $V$.
(2) Evaluate the area of the parallelogram defined by the vectors $U$ and $V$.

Exercise 2. Let $U = (-1, 2, 1)$, $V = (2, 1, 1)$ and

$$S = \{W = (x, y, z) \in \mathbb{R}^3 \mid U \cdot W = 0 \text{ and } V \cdot W = 0\}.$$ 

(1) Show that $S$ is a subspace of $\mathbb{R}^3$.
(2) Find a basis of $S$. 
SOLUTIONS

Ex. 1. The angle between two nonzero vectors \( U, V \) is the real number \( \theta \in [0, \pi] \) such that

\[
\cos(\theta) = \frac{U \cdot V}{||U|| \times ||V||}.
\]

The area of the parallelogram defined by the two vectors is

\[
\mathcal{A}(U, V) = ||U|| \times ||V|| \sin(\theta) \text{ (unit}^2)\]

In our case, \( \cos(\theta) = \frac{1}{\sqrt{2}} \). Hence \( \theta = \frac{\pi}{4} \) and consequently,

\[
\mathcal{A}(U, V) = \sqrt{2} \times \sqrt{4} \times \frac{1}{\sqrt{2}} = 2 \text{ (unit}^2).\]

\[\Box\]

Ex. 2.

1. \(-\) As \( U \cdot 0_{x^3} = V \cdot 0_{x^3} = 0 \), we get \( 0_{x^3} \in S \).
   \(-\) Let \( W_1, W_2 \) be two elements of \( S \). Then

\[
U \cdot W_1 = V \cdot W_1 = U \cdot W_2 = V \cdot W_2 = 0.
\]

This leads to

\[
U \cdot (W_1 + W_2) = U \cdot W_1 + U \cdot W_2 = 0 + 0 = 0,
\]

and

\[
V \cdot (W_1 + W_2) = V \cdot W_1 + V \cdot W_2 = 0 + 0 = 0.
\]

It follows that \( W_1 + W_2 \in S \).
   \(-\) Now, let \( W \in S \) and \( \alpha \in \mathbb{R} \). Then

\[
U \cdot (\alpha W) = \alpha U \cdot W = \alpha \times 0 = 0,
\]

and

\[
V \cdot (\alpha W) = \alpha V \cdot W = \alpha \times 0 = 0.
\]

This implies that \( \alpha W \in S \).

Therefore, \( S \) is a subspace of \( \mathbb{R}^3 \).

2. Let \( W = (x, y, z) \in \mathbb{R}^3 \). Then \( W \in S \) if and only if the following equations are satisfied

\[
-x + 2y - z = 0 \text{ and } 2x + y + z = 0.
\]

This gives \( y = 3x \) and \( z = -5x \). Thus, \( W = (x, y, z) = (x, 3x, -5x) = x(1, 3, -5) \).

We conclude that \( B = \{(1, 3, -5)\} \) is a basis of \( S \). \[\Box\]