1. Suppose $0 < a < b$. Prove that

$$1 - \frac{a}{b} < \ln \frac{b}{a} < \frac{b}{a} - 1.$$  

Use this result to show that

$$\frac{1}{6} < \ln 1.2 < \frac{1}{5}.$$ 

2. Assume that a function $f : [a, b] \to \mathbb{R}$ is twice continuously differentiable. Show that there exists $c \in (a, b)$ such that

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!}f''(c).$$

Hint. Consider the auxiliary function $F$ defined by

$$F(x) = f(b) - f(x) - (b - x)f'(x) - (b - x)^2A,$$

where $A$ is a constant to be determined.

3. Let $f(x) = 1 - (x - 1)^{2/3}$, $0 \leq x \leq 2$. Construct the graph of $f$ and explain why Rolle’s theorem does not apply to this function.

4. Let $I = (1, \infty)$. Consider $f : I \to \mathbb{R}$, defined by $f(x) = (x - 1)^3$. Show that $f$ is invertible and find its inverse function $g$. Verify that

$$g'(x_0) = \frac{1}{f'(g(x_0))}, \quad x_0 \in I.$$ 

5. Show that there exists a number $c \in (0, 1)$ such that

$$\int_0^1 \frac{e^x}{1 + x} dx = e^c \ln 2.$$