1. Let $X$ and $Y$ be two normed linear spaces with $\dim X < +\infty$. Show that any linear operator $A : D(A) = X \to Y$ is bounded and there exists $x_0 \neq 0$ in $X$ such that $\|Ax_0\| = \|A\| \|x_0\|$. 

2. Let $X$ and $Y$ be two Banach spaces and $M$ a subset of $L(X, Y)$ such that $\sup_{A \in M} \|Ax\| < +\infty$ for every $x \in X$. Show that $\sup_{A \in M} \|A\| < +\infty$. 

3. Consider the subspace $M = \{(x, 0) : x \in \mathbb{R}\}$ of $\mathbb{R}^2$ and the transformations $P_i : \mathbb{R}^2 \to M$, $i = 1, 2$, defined by $P_1(x, y) = (x, 0)$ and $P_2(x, y) = (x + \alpha y, 0)$ where $(x, y) \in \mathbb{R}^2$ and $\alpha$ is a scalar. Show that $P_1$ and $P_2$ are projections and $R(P_1) = R(P_2) = M$ despite the fact that $P_1 \neq P_2$. 

4. Let $(X, d)$ be a complete metric space and let $f : X \to X$ be a contraction mapping with a constant $k$. If $x^* \in X$ is the fixed point of $f$, then show that 

$$d(x^*, x) \leq \frac{1}{1-k} d(f(x), x) \quad \text{for any } x \in X,$$

and $x^*$ is a fixed point of the function $f^n$ defined as $f^n = f \circ f^{n-1}$, $f^1 = f$, $n$ is a positive integer. 

5. Let $X = C^1([a, b] ; \mathbb{R})$. Define a function $N$ on $X$ by 

$$N(f) = \sqrt{f(a)^2 + \int_a^b f'(x)^2 dx}.$$

Show that this function defines a norm on $X$. 

6. Let $U$ be a normed space, $f \in U'$ and $u_0 \notin N(f)$. Show that there is a vector $w \in N(f)$ such that 

$$\|w - u_0\| = d(u_0, N(f))$$

if and only if there exists a vector $u \in U$ such that $\|u\| = 1$ and $f(u) = \|f\|$. 
