King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Midterm Exam for MATH 572, Semester 101

Problem 1 Consider the following one dimensional parabolic problem

$$\begin{cases} u_t - u_{xx} = f \quad \text{in } (0,1)\times(0,T) \\ u_x(0,t) = u_x(1,t) = 0 \quad \text{for } t \in (0,T), \\ u(x,0) = v(x) \quad \text{for } x \in (0,1), \end{cases} \tag{1}$$

where $u = u(x,t)$ and $f = f(x,t)$. Assume that f and v are sufficiently regular

a) Define the spatial piecewise linear FE solution u_h of (1) (assume that $u_h(0) = v_h \approx v$)

b) Derive the following stability property of u_h:

$$\|u_h(t)\| \le \|v_h\| + \int_0^t \|f(x,s)\|\,ds \quad \text{for each } t \in (0,T]$$

c) Show that for each $t \in (0,T]$, the FE solution u_h exists and unique.

Problem 2 Consider the following singularly perturbed two-point BVP:

$$\begin{cases} -\varepsilon u'' + u = f \quad \text{in } (0,1) \\ u'(0) + u(0) = 0 \quad \text{and} \quad u(1) = 0, \end{cases} \tag{2}$$

where $u = u(x)$, $f = f(x)$, and ε is a strictly positive constant. Assume that f is sufficiently regular.

a) Define the weak formulation of (2)

b) Show that (2) has a unique weak solution

c) Define the piecewise linear FE solution u_h of (2)

d) Prove that u_h exists and unique

e) Derive the error $\|u - u_h\|_1$

f) Justify that for ε sufficiently small, the order of convergence of the error $\|u - u_h\|_1$ need not be one.

Problem 3 Consider the following two-point BVP:

$$\begin{cases} -u'' + bu = f \quad \text{in } (0,1) \\ u'(0) = u'(1) = 0, \end{cases} \tag{3}$$

where $u = u(x)$ and $f = f(x)$. Assume that u is the strong solution of (3).

a) If $b \ge b_0 > 0$, show that $\|u\|_2 \le C\|f\|$.

b) Justify that the inequality: $\|u\|_2 \le C\|f\|$ may not be applicable to (3) when $b = 0$.