King Fahd University of Petroleum and Minerals
Quiz 3 Math 102, Semester 102

Name: ID: Section:

Note: Justify your answers in all the questions below.

Q 1) Determine whether the series $\sum_{n=0}^{\infty} \frac{\sin(n\pi)+(-1)^n 3^{n-1}}{2^{2n+1}}$ is convergent or divergent. If it is convergent, find its sum.

solution : $\sin(n\pi) = 0$, so $\sum_{n=0}^{\infty} \frac{\sin(n\pi)+(-1)^n 3^{n-1}}{2^{2n+1}}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-1} . 3^n}{4^n . 2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{6} \cdot \left(\frac{-3}{4}\right)^n = \underbrace{\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{-3}{4}\right)^{n-1}}_{\text{Geometric series}, \ a=\frac{1}{6}, \ r=-\frac{3}{4}}$$

$|r| = \frac{3}{4} < 1$ so, it is convergent.

and $\sum_{n=0}^{\infty} \frac{\sin(n\pi)+(-1)^n 3^{n-1}}{2^{2n+1}} = \frac{1}{6} \cdot \frac{1}{1-(-\frac{3}{4})} = \frac{1}{6} \cdot \frac{1}{1+\frac{3}{4}}$

$$= \frac{2}{21}$$

Q2) Determine whether the following sequences are convergent or divergent

a) $\left\{\sin\left(\frac{n\pi}{2}\right)\frac{\sqrt{n}}{n+1}\right\}_{n=1}^{\infty}$

b) $\left\{\frac{1-2\cos(n)}{\ln(n)}\right\}_{n=2}^{\infty}$.

a) $a_n = \sin\left(\frac{n\pi}{2}\right)\frac{\sqrt{n}}{n+1}$, $\boxed{\frac{-\sqrt{n}}{n+1}} \leq a_n \leq \boxed{\frac{\sqrt{n}}{n+1}}$

$\frac{-\sqrt{n}}{n+1} \xrightarrow{n\to\infty} 0$, $\frac{\sqrt{n}}{n+1} \xrightarrow{n\to\infty} 0$

$\therefore$, by squeeze theorem, $\lim_{n\to\infty} a_n = 0$

& thus, $\left\{\sin\left(\frac{n\pi}{2}\right)\frac{\sqrt{n}}{n+1}\right\}_{n=1}^{\infty}$ is convergent.

b) $a_n = \dfrac{1-2\cos(n)}{\ln(n)}$

$-1 \leq \cos(n) \leq 1 \Rightarrow$ $-2 \leqslant -2\cos(n) \leqslant 2$

$$\Downarrow$$

$$-1 \leqslant 1-2\cos(n) \leqslant 3$$

$$\Downarrow$$

$$\frac{-1}{\ln(n)} \leqslant \frac{1-2\cos(n)}{\ln(n)} \leqslant \frac{3}{\ln(n)}$$

$$\frac{-1}{\ln(n)} \xrightarrow{n\to\infty} 0 \qquad \frac{3}{\ln(n)} \xrightarrow{n\to\infty} 0$$

$\therefore$, $\lim\limits_{n\to\infty} \dfrac{1-2\cos(n)}{\ln(n)} = 0$ (by squeeze theorem)

and hence, $\left\{ \dfrac{1-2\cos(n)}{\ln(n)} \right\}_{n=2}^{\infty}$ is convergent.

Q 3) Find the sum of the series: $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$

$n^2+3n+2 = (n+1)(n+2)$, so,

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+3n+2)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=0}^{\infty}\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \sum_{n=1}^{\infty}\underbrace{\left[\frac{1}{n} - \frac{1}{n+1}\right]}_{a_n}$$

$$S_n = \sum_{i=1}^{n} a_i$$

$$= \left\{ 1 - \frac{1}{n+1} \right.$$

$$\lim_{n\to\infty} S_n = 1 \quad \therefore \quad \sum_{n=0}^{\infty} \frac{1}{(n^2+3n+2)} = \lim_{n\to\infty} S_n = 1 .$$

Q 4) Test the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ for convergence or divergence.

$a_n = \frac{1}{n\ln(n)}$, a_n is continuous, positive & decreasing on $[2, \infty)$.

Let $f(x) = \frac{1}{x\ln x}$,

$$\int_2^t \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\ln t} \frac{du}{u} = \ln(\ln t) - \ln(\ln 2) \longrightarrow \infty \text{ as } t\to\infty$$

$u = \ln x \Rightarrow du = \frac{dx}{x}$

$\therefore \int_2^{\infty} \frac{1}{x\ln x}\,dx$ is divergent and then $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is divergent by the integral test.