Math 202- Elements of Differential Equations
Major Exam I- Semester 102 (2010-2011)
King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Duration: 2h

Name: Instructors Key Solution
ID: ————————————
Section: All

• Write all steps clearly. Otherwise points might be deducted.
• Calculators and mobiles are not allowed in this exam.

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1
Exercise 1 (11 Points)

Consider the following differential equation:

\[(dy/dx - y)^2(d^2y/dx^2 + y)^2 = 0\]

a) Find the order of this differential equation.

b) Is it linear or nonlinear? Justify your answer.

c) Find the value(s) of \(m\) for which the function \(y = e^{mx}\) is a solution.

Solution

a) The differential equation could be written:

\[(dy/dx - y)(d^2y/dx^2 + y) = 0 \Rightarrow 1 \text{ point}\]

Which is also \((y' - y)(y'' + y) = 0 \Rightarrow 1 \text{ point}\)

or \((y' - y)y'' + yy' + y^2 = 0 \Rightarrow \text{ or } 1 \text{ point}\)

This is a second order differential equation. \(\Rightarrow 1 \text{ point}\)

Maximum grade here is 3 points.

b) The differential equation is nonlinear because the coefficients of \(y\), \(y'\) and \(y''\) depend on \(y\) and its derivatives.

Just give 1 point if the answer is not justified.

Maximum grade here is 3 points.

c) If \(y = e^{mx}\), then \(y' = me^{mx}\) and \(y'' = m^2e^{mx}\) \(\Rightarrow 1 \text{ point}\).

Substitution yields

\((y' - y)(y'' + y) = 0 \Leftrightarrow (me^{mx} - e^{mx})(m^2e^{mx} + e^{mx}) = 0 \Rightarrow 1 \text{ point}\).

\(e^{mx}\) can be eliminated because \(e^{mx} \neq 0\) for all \(x \in \mathbb{R}\), the remaining equation is

\((m - 1)(m^2 + 1) = 0. \Rightarrow 1 \text{ point}\).

Thus \(m = 1 \Rightarrow 1 \text{ point}\),

or \(m = \pm i. \Rightarrow 1 \text{ point}\).

Just give 4 points if the \(m = \pm i\) is missing but the answer is complete.

1 point should subtracted for each missing step.

Maximum grade here is 5 points.
Exercise 2 (11 Points)

Determine the largest region of the $xy$-plane for which the following differential equation would have a unique solution whose graph passes through the point $(x_0; y_0)$.

$$(y^2 - 4)y' = (x^2 + 4) \ln(y)$$

Solution

The differential equation could be written

$$y' = (x^2 + 4) \frac{\ln(y)}{(y^2 - 4)}. \quad \Rightarrow \text{2 points}$$

Consider

$$f(x, y) = (x^2 + 4) \frac{\ln(y)}{(y^2 - 4)},$$

then $f$ is defined and continuous for $x \in \mathbb{R}$ and $y \in (0; 2) \cup (2; \infty)$, $\Rightarrow$ 2 points.

and $\frac{df}{dy} = (x^2 + 4) \left( \frac{1}{y(y^2 - 4)} - \frac{2y \ln(y)}{(y^2 - 4)^2} \right)$. $\Rightarrow$ 2 points.

Thus $\frac{df}{dy}$ is defined and continuous also for $x \in \mathbb{R}$ and $y \in (0; 2) \cup (2; \infty)$. $\Rightarrow$ 2 points.

The largest region of the $xy$-plane is such that $y \in (2; \infty)$. $\Rightarrow$ 3 points

You should give 11 points only to a complete and detailed answer.
Exercise 3 (11 Points)

Consider the initial value problem for the differential equation:

\[ ye^y(x^3 - 7)dy + x^2 e^{2y}dx = 0 \]
\[ y(2) = 0 \]

a) Find an implicit solution to this problem.
b) What is the largest interval of definition of the solution?

Solution

a) The differential equation could be written:

\[ ye^y(x^3 - 7)dy = -x^2 e^{2y}dx \Leftrightarrow ye^{-y}dy = -\frac{x^2}{x^3-7}dx. \Rightarrow 1 \text{ point} \]

The differential equation is now clearly separable. \Rightarrow 1 \text{ point} 

We integrate both parts: \[ \int ye^{-y}dy = \int -\frac{x^2}{x^3-7}dx. \Rightarrow 1 \text{ point} \]

By part integration yields:

\[ \int ye^{-y}dy = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y}. \Rightarrow 2 \text{ points} \]

The right side would be also integrated:

\[ \int -\frac{x^2}{x^3-7}dx = -\frac{1}{3} \ln|x^3 - 7| + C \Rightarrow 2 \text{ points} \]

The implicit solution is then:

\[ -e^{-y}(y + 1) = -\frac{1}{3} \ln|x^3 - 7| + C. \Rightarrow 1 \text{ point} \]

Since \( y(2) = 0 \), we should have \( -e^{0}(0 + 1) = -\frac{1}{3} \ln(2^3 - 7) + C. \)

Thus \( C = -1. \Rightarrow 1 \text{ point} \)

The maximum grade here is 9 points.

b) The solution is defined if \( x^3 - 7 > 0 \Rightarrow x > 7^{\frac{1}{3}}. \Rightarrow 1 \text{ point} \)

The largest interval of definition of the solution is then \( (7^{\frac{1}{3}} ; \infty) \Rightarrow 1 \text{ point} \)

The maximum grade here is 2 points.
Exercise 4 (11 Points)
Consider the following differential equation
\[(xy^3 + y)dx + (2x^2y^2 + 2x + 2y^4)dy = 0.\]
a) Find an integrating factor that makes the differential equation exact.
b) Solve the exact differential equation obtained in (a).

Solution

a) We consider \(M(x,y) = xy^3 + y\), then \(M_y = \frac{dM}{dy} = 3xy^2 + 1\). \(\Rightarrow 1\) point
And \(N(x,y) = 2x^2y^2 + 2x + 2y^4\), then \(N_x = \frac{dN}{dx} = 4xy^2 + 2\). \(\Rightarrow 1\) point
Since \(M_y \neq N_x\), the equation is not exact. We can try to find a single variable integrating factor:
We first try \(\frac{M_y-N_x}{N} = \frac{-xy^2-1}{2x^2y^2+2x+2y^4}\). It does not depend only on \(x\). \(\Rightarrow 1\) point
We try \(\frac{N_x-M_y}{M} = \frac{xy^2+1}{xy^4+y} = \frac{1}{y}\). It depends only on \(y\). \(\Rightarrow 1\) point
The integrating factor to be used is then \(e^{\int \frac{1}{y} dy} = y\). \(\Rightarrow 1\) point

The maximum grade here is 5 points.

b) We now multiply the differential equation by \(y\).
It becomes \((xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0\). \(\Rightarrow 1\) point
The differential equation is now exact. We do not need to prove it again.
Let us consider
\[F(x,y) = \int (xy^4 + y^2)dx + g(y) \Rightarrow F(x,y) = \frac{1}{2}x^2y^4 + xy^2 + c_1 + g(y). \Rightarrow 2\) points
Once we derivate \(F\) with respect to \(y\) we get:
\[\frac{dF}{dy} = 2x^2y^3 + 2xy + g'(y) = (2x^2y^3 + 2xy + 2y^5) \Rightarrow 1\) point
thus \(g'(y) = 2y^5 \Rightarrow g(y) = \frac{1}{3}y^6 + c_2. \Rightarrow 1\) point
The solution of the differential equation is then
\[F(x,y) = \frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = C. \Rightarrow 1\) point
\(C\) collects all integration constants here.

The maximum grade here is 6 points.
Exercise 5 (14 Points)
Solve the following differential equations by substitution.

(a) $x^2y' + 2xy = 5y^3$.
(b) $xy' = y + \sqrt{x^2 + y^2}$.

Solution

(a)

(i) We divide the differential equation by $x^2$, we get $y' + \frac{2}{x}y = 5y^3$.

This is a Bernoulli differential equation with $n = 3$.

The substitution to be used is $u = y^{1-n} = y^{-2}$. Thus $y = u^{-\frac{1}{2}} \Rightarrow 1$ point

If we derivate $y' = \frac{dy}{dx} = -\frac{1}{2} \frac{du}{dx} u^{-\frac{3}{2}} \Rightarrow 1$ point

Then we substitute in the differential equation and we get

$-\frac{1}{2}u^{-\frac{3}{2}} \frac{du}{dx} + \frac{2}{x}u^{-\frac{3}{2}} = 5u^{-\frac{3}{2}}$.

Which is equivalent to $-\frac{1}{2} \frac{du}{dx} + \frac{2}{x}u = \frac{5}{x^2}$.

We now find a linear first order differential equation:

$\frac{du}{dx} - \frac{4}{x}u = -\frac{10}{x^2} \Rightarrow 1$ point

Up to here you should give 3 points only to a complete and detailed answer.

(ii) Let $P(x) = -\frac{4}{x}$ so the integrating factor would be $e^{\int -\frac{4}{x}dx} = x^{-4} \Rightarrow 1$ point

Thus $\frac{d}{dx}(ux^{-4}) = -\frac{10}{x^2}$.

$u(x) = x^4 \int -\frac{10}{x^2}dx \Rightarrow u(x) = x^4(2x^{-5} + c) = (2x^{-1} + cx^4) \Rightarrow 1$ point

Finally $y(x) = (2x^{-1} + cx^4)^{-\frac{1}{2}}$, where $c$ is an arbitrary constant. $\Rightarrow 1$ point

The maximum grade here is 6 points.

See next page for question (b).
(b) (i) The differential equation could be written
\[ x \frac{dy}{dx} = y + \sqrt{x^2 + y^2} \Leftrightarrow xdy = (y + \sqrt{x^2 + y^2})dx. \]
Which yields \(- (y + \sqrt{x^2 + y^2})dx + xdy = 0. \Rightarrow 1 \text{ point}\)

(ii) Let \(M(x, y) = -(y + \sqrt{x^2 + y^2}),\)
then \(M(tx, ty) = -(ty + \sqrt{(tx)^2 + (ty)^2}) = tM(x, y). \Rightarrow 1 \text{ point}\)
The same way let \(N(x, y) = x,\)
then \(N(tx, ty) = tN(x, y). \Rightarrow 1 \text{ point}\)
Clearly both functions \(M\) and \(N\) are homogeneous of degree 1.

(iii) We can set \(u = \frac{y}{x}\) \((x \neq 0),\) so \(y = ux\) and \(dy = udx + xdu. \Rightarrow 1 \text{ point}\)
Give just 0.5 if \(dy = udx + xdu\) is missing, and 0 if the substitution is wrong.

Up to here you should give 4 points only to a complete and detailed answer in steps (i), (ii) and (iii).

(iv) After substitution the differential equation becomes
\[- (ux + \sqrt{x^2 + x^2u^2})dx + x(udx + xdu) = 0.\]
This is equivalent to \(- \sqrt{x^2} \sqrt{1 + u^2}dx + x^2du = 0. \Rightarrow 1 \text{ point}\)
The differential equation becomes separable \(\frac{1}{\sqrt{1+u^2}}du = \frac{1}{\sqrt{x^2}}dx, \text{ if } x \neq 0.\)
Thus \(\int \frac{1}{\sqrt{1+u^2}}du = \int \frac{1}{\sqrt{x^2}}dx. \Rightarrow 1 \text{ point}\
We have \(\int \frac{1}{\sqrt{1+u^2}}du = \int \sec \theta d\theta, \text{ with } u = \tan \theta.\)
We know that \(\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|, \text{ thus}\)
\(\int \frac{1}{\sqrt{1+u^2}}du = \ln |\sqrt{1 + u^2} + u|. \Rightarrow 1 \text{ point}\)
On the right side we will have \(\int \frac{1}{\sqrt{x^2}}dx = (\text{sign } x) \ln |x| + C.\)
Thus \(\ln |\sqrt{1 + u^2} + u| = (\text{sign } x) \ln |x| + C.\)
Finally \(\ln |\sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x}| = (\text{sign } x) \ln |x| + C. \Rightarrow 1 \text{ point}\)
Subtract 0.5 point if \((\text{sign } x)\) is missing.
You should give 4 points only to a complete and detailed answer in step (iv).

The maximum grade here is 8 points.
Exercise 6 (11 Points)

A thermometer is taken from an inside room to the outside, where the air temperature is 20°C. After 1 minute the thermometer reads 35°C, and after 4 minutes it reads 27°C. What is the initial temperature of the inside room?

Solution

Newton’s law of warming/cooling is: \( \frac{dT}{dt} = k(T - A) \), where \( A \) is the air temperature in this example. ⇒ 1 point

Thus \( \frac{dT}{T - A} = kdt \) ⇒ \( T(t) = A + \alpha e^{kt} \). ⇒ 2 points

We know that \( A = 20 \) and at \( t = 1 \), \( T(1) = 35 \).
Thus \( 20 + \alpha e^k = 35 \) ⇒ \( \alpha e^k = 15 \). ⇒ 1 point

At \( t = 4 \), \( T(4) = 27 \). Thus \( 20 + \alpha e^{4k} = 27 \) ⇒ \( \alpha e^{4k} = 7 \). ⇒ 1 point

Hence \( e^{3k} = \frac{7}{15} \) ⇒ \( k = \frac{1}{3} \ln \left( \frac{7}{15} \right) \). ⇒ 1 point

And \( \alpha = 15e^{-\frac{1}{3} \ln \left( \frac{7}{15} \right)} \). ⇒ 1 point

The solution is then \( T(t) = 20 + 15e^{-\frac{1}{3} \ln \left( \frac{7}{15} \right)}e^{\frac{1}{3} \ln \left( \frac{7}{15} \right)t} \). ⇒ 2 points

At \( t = 0 \), the temperature of the thermometer is equal to the temperature of the inside room \( T(0) = 20 + 15e^{-\frac{1}{3} \ln \left( \frac{7}{15} \right)} \). ⇒ 2 points

You should give 11 points only to a complete and detailed answer.

The maximum grade here is 11 points.
Exercise 7 (11 Points)
Consider the third-order differential equation
\[ y''' - 5y'' + 8y' - 4y = 0 \]
a) Verify that \( y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \) is a family of solutions.
b) Find a member of this family of solutions that satisfies the initial conditions
\[ y(0) = 1, \quad y'(0) = 4, \quad \text{and} \quad y''(0) = 0. \]

Solution
a) Let \( y_1 = e^x \), then \( y_1' = y_1'' = y_1''' = e^x. \) ⇒ 1 point
It is clear that \( e^x - 5e^x + 8e^x - 4e^x = 0. \) \( y_1 = e^x \) is a solution. ⇒ 1 point

Let \( y_2 = e^{2x} \), then \( y_2' = 2e^{2x}, \ y_2'' = 4e^{2x}, \ y_2''' = 8e^{2x}. \) ⇒ 1 point
It is clear that \( 8e^{2x} - 5 \times 4e^{2x} + 8 \times 2e^{2x} - 4e^{2x} = 0. \) \( y_2 = e^{2x} \) is a solution. ⇒ 1 point

Let \( y_3 = x e^{2x} \), then \( y_3' = 2xe^{2x} + e^{2x}, \ y_3'' = 4xe^{2x} + 4e^{2x}, \ y_3''' = 8xe^{2x} + 12e^{2x}. \) ⇒ 1 point
It is clear that \( 8xe^{2x} + 12e^{2x} - 5 \times 4xe^{2x} - 5 \times 4e^{2x} + 8 \times 2xe^{2x} + 8e^{2x} - 4xe^{2x} = 0. \) \( y_2 = e^{2x} \) is a solution. ⇒ 1 point
Thus \( y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \) is a family of solutions.
You should give 6 points only to a complete and detailed answer.

b)
We have \( y(0) = 1, \) then \( c_1 + c_2 = 1. \) ⇒ 1 point

We have \( y'(x) = c_1 e^x + 2c_2 e^{2x} + c_3 (2xe^{2x} + e^{2x}). \)
If \( y'(0) = 4, \) then \( c_1 + 2c_2 + c_3 = 4. \) ⇒ 1 point

We have \( y''(x) = c_1 e^x + 4c_2 e^{2x} + c_3 (4xe^{2x} + 4e^{2x}). \)
If \( y''(0) = 0, \) then \( c_1 + 4c_2 + 4c_3 = 0. \) ⇒ 1 point
You should give 1 point for the linear system solution procedure if it is clear. You can accept any method.

The solution here $c_1 = -12, c_2 = 13, c_3 = -10$.
Thus $y(x) = -12e^x + 13e^{2x} - 10xe^{2x}$ is the solution satisfying the conditions. ⇒ 1 point

You should give 5 points only to a complete and detailed answer.
Exercise 8 (8 Points)
a) Find the Wronskian of \( f(x) = x^3, \ g(x) = x^2 \) and \( h(x) = \ln(x) \).
b) Give the largest interval in which these functions are linearly independent.
solution

a) The Wronskian corresponding to these functions is

\[
W = \begin{vmatrix} x^3 & x^2 & \ln x \\ 3x^2 & 2x & 1/x \\ 6x & 2 & -1/x^2 \end{vmatrix}
\]

\( \Rightarrow 3 \) points

Thus \( W = x^3(-\frac{4}{x}) - 3x^2(-1 - 2 \ln x) + 6x(x - 2x \ln x). \Rightarrow 1 \) point

\( \Rightarrow W = -4x^2 + 3x^2 + 6x^2 \ln x + 6x^2 - 12x^2 \ln x. \)

\( \Rightarrow W = 5x^2 - 6x^2 \ln x = x^2(5 - 6 \ln x). \Rightarrow 1 \) point

You should give 5 points only to a complete and detailed answer.

b) These function are linearly independent if \( W \neq 0. \)

Thus \( x \neq 0, \Rightarrow 1 \) point

and \( x \neq e^{\frac{5}{6}}. \Rightarrow 1 \) point

The largest interval where these functions are independent is \( (e^{\frac{5}{6}}, \infty). \Rightarrow 1 \) point

You should give 3 points only to a complete and detailed answer.