Math 202- Elements of Differential Equations
Major Exam II- Semester 102 (2010-2011)
King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Duration: 2h

Instructors Solution Key
All Sections

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Exercise 1 (16 points)

a.) Find $A$, $B$ and $C$ such that $y = Ax^2 + Bx + C$ is a solution to the differential equation
\[ y'' + 4y' - y = -x^2 + 1. \]

b.) Find $D$ and $E$ such that $y = (Dx + E)e^{-x}$ is a solution to the differential equation
\[ y'' + 4y' - y = xe^{-x} - e^{-x}. \]

c.) Find a particular to the following differential equation
\[ y'' + 4y' - y = -x^2 + 1 + xe^{-x} - e^{-x}. \]

Solution

a) $y = Ax^2 + Bx + C$, $y' = 2Ax + B$, $y'' = 2A$. $\Rightarrow 1$ point

Then
\[ y'' + 4y' - y = 2A + 4(2Ax + B) - (Ax^2 + Bx + C) = -Ax^2 + (8A - B)x + 2A + 4B - C. \]
\[ = -x^2 + 1. \]

$\Rightarrow 2$ points

By identification we have
\[
\begin{aligned}
-A &= -1 \\
8A - B &= 0 \\
2A + 4B - C &= 1
\end{aligned}
\]

Thus $A = 1$, $B = 8$ and $C = 33$.

$\Rightarrow 3$ points

1.5 points if the system is correct but $A$, $B$ and $C$ are wrong.

The maximum number of points here is 6.

b) $y = (Dx + E)e^{-x}$, $y' = -(Dx - E + D)e^{-x}$, $y'' = (Dx + E - 2D)e^{-x}$. $\Rightarrow 2$ points

Then
\[ y'' + 4y' - y = (Dx + E - 2D + 4(-Dx - E + D) - (Dx + E))e^{-x} = (-4Dx + 4E + 2D)e^{-x}. \]

By identification we have $-4D = 1$ and $2D - 4E = -1$. Thus $D = -\frac{1}{4}$ and $E = \frac{1}{8}$.

$\Rightarrow 2$ points

The maximum number of points here is 6.

c) From a) and b) we deduce that $y_p = x^2 + 8x + 33 - \frac{1}{4}xe^{-x} + \frac{1}{8}e^{-x}$.

$\Rightarrow 2$ points for each part.

Give 4 points if the student solves the equation without using a) and b) and gets the correct answer.

The maximum number of points here is 4.
Exercise 2 (14 points)

a) Verify that \( y_1 = x \) is a solution of

\[
(1 + x^2)y'' - 2xy' + 2y = 0.
\]

b) Find the general solution of the differential equation.

Solution

a) \( y_1 = x, \ y_1' = 1, \ y_1'' = 0. \)

\[
(1 + x^2)y_1'' - 2xy_1' + 2y_1 = -2x + 2x = 0.
\]

Give 0 or 4 for this question; No partial credit if the students did not get the correct answer.

The maximum number of points here is 4.

b) We first write the differential equation in its standard form

\[
y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0. \quad \Rightarrow 2 \text{ points}
\]

Let \( y_2 \) be the second solution, then

\[
y_2 = y_1 \int \frac{e^{-\int \frac{P(x)dx}{y_1^2}}} \Rightarrow 2 \text{ points}
\]

\[
y_2 = x \int \frac{\frac{2x}{1+x^2}dx} + x \int \frac{\ln(1+x^2)}{x^2}dx = x \int \frac{1+x^2}{x^2}dx
\]

\( \Rightarrow 2 \text{ points} \) (1+1 for the two last steps, 0 if the student did not integrate.)

Thus \( y_2 = x \int (1 + \frac{1}{x^2})dx = x(x - \frac{1}{x}) = x^2 - 1. \)

\( \Rightarrow 2 \text{ points} \) (Penalize (-1) just in one place if the student writes \( y_2 = x^2 + 1. \))

The general solution is \( y = c_1y_1 + c_2y_2 = c_1x + c_2(x^2 - 1), \Rightarrow 2 \text{ points} \)

with \( c \) arbitrary constants.

If the student did not use the standard form: Give 2 points if he wrote the correct formula for \( y_2 \) and add 1 point if he wrote the form of the general solution correctly.

The maximum number of points here is 10.
Exercise 3 (14 points)

What is the general solution of

\[(D^2 - 4)(D - 2)^2(8D^2 + 4D + 1)y = 0\]

Solution

First Approach:

\[D^2 - 4 = (D + 2)(D - 2)\] Thus \(e^{2x}\) and \(e^{-2x}\) are solutions. \(\Rightarrow 3\) points

\((D - 2)^2\) suggests that \(xe^{2x}\) and \(x^2e^{2x}\) are solutions. \(\Rightarrow 4\) points

For \(8D^2 + 4D + 1\) the roots of the auxiliary equation are \(-\frac{1}{4} \pm \frac{1}{4}i\). \(\Rightarrow 3\) points

Hence \(e^{-\frac{x}{4}}cos(\frac{x}{4})\) and \(e^{-\frac{x}{4}}sin(\frac{x}{4})\) are solutions. \(\Rightarrow 1\) point

The general solution is

\[y = c_1e^{-2x} + c_2e^{2x} + c_3xe^{2x} + c_4x^2e^{2x} + c_5e^{-\frac{x}{4}}cos(\frac{x}{4}) + c_6e^{-\frac{x}{4}}sin(\frac{x}{4}), \Rightarrow 3\] points

with \(c\) arbitrary constants.

The maximum number of points here is 14.

Second Approach:

The auxiliary equation is: \((m^2 - 4)(m - 2)^2(8m^2 + 4m + 1) = 0\) \(\Rightarrow 2\) points

The roots are \(-2, 2\) repeated 3 times and \(-\frac{1}{4} \pm \frac{1}{4}i\) \(\Rightarrow 3\) points, subtract 1 point if the student did not give the conjugate complex root.

The solutions are \(e^{-2x}, e^{2x}, xe^{2x}, x^2e^{2x}, e^{-\frac{x}{4}}cos(\frac{x}{4})\) and \(e^{-\frac{x}{4}}sin(\frac{x}{4})\). \(\Rightarrow 6\) points, 1 point for each solution.

The general solution is

\[y = c_1e^{-2x} + c_2e^{2x} + c_3xe^{2x} + c_4x^2e^{2x} + c_5e^{-\frac{x}{4}}cos(\frac{x}{4}) + c_6e^{-\frac{x}{4}}sin(\frac{x}{4}), \Rightarrow 3\] points

with \(c\) arbitrary constants.

The maximum number of points here is 14.

Third Approach:

If the student gives directly the general solution without details: give full mark if the answer is correct OR follow the point distribution for each solution if the answer is not correct.
Exercise 4 (14 points)

Determine the form of a general solution for

\[ y''' - 2y'' = 5x - x^3 e^{2x} + x^2 e^{-x} \sin(2x) \]

(Do not find the constants values.)

Solution

The differential equation could be written: \( D^2(D-2)y = 5x - x^3 e^{2x} + x^2 e^{-x} \sin(2x) \).

Thus \( y_c(x) = a + bx + ce^{2x} \), \( a, b \) and \( c \) are arbitrary constants. ⇒ 3 points

The annihilators of the right hand side are:

- \( 5x \) ⇒ \( D^2 \). ⇒ 1 point
- \( x^3 e^{2x} \) ⇒ \( (D - 2)^4 \). ⇒ 2 points
- \( x^2 e^{-x} \sin(2x) \) ⇒ \( (D^2 + 2D + 5)^3 \). ⇒ 3 points

Thus \( (D^2 + 2D + 5)^3 D^2(D-2)^5 y = 0 \).

A particular solution would have the form:

\[
y_p(x) = c_1 x^2 + c_2 x^3 + c_3 x e^{2x} + c_4 x^2 e^{2x} + c_5 x^3 e^{2x} + c_6 x^4 e^{2x} + c_7 e^{-x} \cos(2x) + c_8 e^{-x} \sin(2x) + c_9 x e^{-x} \cos(2x) + c_{10} x e^{-x} \sin(2x) + c_{11} x^2 e^{-x} \cos(2x) + c_{12} x^2 e^{-x} \sin(2x).\]

⇒ 4 points

Thus the general solution could be written

\[
y(x) = a + bx + ce^{2x} + c_1 x^2 + c_2 x^3 + c_3 x e^{2x} + c_4 x^2 e^{2x} + c_5 x^3 e^{2x} + c_6 x^4 e^{2x} + c_7 e^{-x} \cos(2x) + c_8 e^{-x} \sin(2x) + c_9 x e^{-x} \cos(2x) + c_{10} x e^{-x} \sin(2x) + c_{11} x^2 e^{-x} \cos(2x) + c_{12} x^2 e^{-x} \sin(2x).\]

⇒ 1 point (also give this 1 point if the students just writes \( y = y_c + y_p \)).

The maximum number of points here is 14.
Exercise 5 (14 points)
Find a particular solution of the differential equation:

\[ 4y'' + 16y = \csc 2x. \]

Solution

We first write the standard form

\[ y'' + 4y = \frac{1}{4} \csc 2x. \Rightarrow 1 \text{ point} \]

The auxiliary equation of the homogeneous equation is \( m^2 + 4 = 0. \Rightarrow 1 \text{ point} \)
Thus \( y_c(x) = c_1 \cos(2x) + c_2 \sin(2x), \Rightarrow 2 \text{ point} \)

with \( c \) arbitrary constants.

We use the variation of parameters method to find \( y_p \).

\[
\begin{vmatrix}
\cos(2x) & \sin(2x) \\
-2\sin(2x) & 2\cos(2x)
\end{vmatrix} = 2 \neq 0. \Rightarrow 2 \text{ points}
\]

Let \( y_p = u_1 y_1 + u_2 y_2 \), then

\[
u_1' = \frac{1}{2} \begin{vmatrix}
0 & \sin(2x) \\
\frac{1}{4} \csc(2x) & 2\cos(2x)
\end{vmatrix} = \frac{1}{8}(-\sin(2x) \csc(2x)) = -\frac{1}{8}.
\Rightarrow 2 \text{ points, (1 point for } w_1 \text{ and 1 point for } u_1').
\]

And

\[
u_2' = \frac{1}{2} \begin{vmatrix}
\cos(2x) & 0 \\
-2\sin(2x) & \frac{1}{4} \csc(2x)
\end{vmatrix} = \frac{1}{8}(\cos(2x) \csc(2x)) = \frac{\cos(2x)}{8 \sin(2x)}.
\Rightarrow 2 \text{ points, (1 point for } w_2 \text{ and 1 point for } u_2').
\]

Integrating \( u_1' \) and \( u_2' \) yields:

\( u_1 = -\frac{1}{8}x \Rightarrow 1 \text{ point} \)

and \( u_2 = \frac{1}{16} \ln |\sin(2x)|. \Rightarrow 2 \text{ points} \)

Hence

\[ y_p(x) = -\frac{1}{8}x \cos(2x) + \frac{1}{16} \sin(2x) \ln |\sin(2x)|. \Rightarrow 1 \text{ point} \]

If the student did not write the standard form: Penalize 3 points;

\( \Rightarrow 1 \text{ point for the standard form, 1 point for } w_1 \text{ and 1 point } w_2. \)

The maximum number of points here is 14.
Exercise 6 (16 points)
a) Solve the Cauchy-Euler differential equation: \( x^2y'' - xy' + 2y = 0, \ x \in (0, \infty). \)

b) Convert the following equation to a Non-Homogeneous linear differential equation with constant coefficients

\[ x^2y'' - 2xy' + 2y = \ln(x), \ x \in (0, \infty). \]

(Do not solve the differential equation.)

Solution

a) Let \( y(x) = x^m, \ y'(x) = mx^{m-1} \) and \( y'' = m(m-1)x^{m-2}. \)

The auxiliary equation is \( m^2 - 2m + 2 = 0. \) \( \Rightarrow 4 \) points

Thus \( \Delta = -4 \) and \( m = 1 \pm i. \) \( \Rightarrow 2 \) points

Hence \( y(x) = c_1 x \cos(\ln x) + c_2 x \sin(\ln x), \) \( \Rightarrow 2 \) points

with \( c \) arbitrary constants.

The maximum number of points here is 8.

b) To convert the equation to a Non-Homogeneous linear differential equation with constant coefficients we assume that \( x = e^t \) or \( t = \ln x. \) \( \Rightarrow 1 \) point

It follows that

\[ \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}. \] \( \Rightarrow 2 \) points

And \( \frac{d^2y}{dx^2} = \frac{1}{x} \frac{d}{dx} \left( \frac{dy}{dt} \right) + \frac{dy}{dt} \left( -\frac{1}{x^2} \right) = \frac{1}{x^2} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right). \) \( \Rightarrow 3 \) points

Substitution yields

\[ x^2 \frac{1}{x^2} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right) - 2x \frac{1}{x} \frac{dy}{dt} + 2y = t. \]

\( \Rightarrow \)

\[ \frac{d^2y}{dt^2} - 2 \frac{dy}{dt} + 2y = t, \ t \in (-\infty, \infty). \] \( \Rightarrow 2 \) points

Subtract 0.5 point if the student did not write the interval of \( t. \)

The maximum number of points here is 8.