Q1) Show that  \[ \int_{0}^{\infty} \frac{x^{a-1}}{1-x} \, dx = \pi \cot(\alpha \pi), \quad 0 < \alpha < 1. \]

Q2) Find the inverse transforms as indicated

(a) \[ L^{-1} \left\{ \frac{(s+1)}{(s+2)(s+3)} \right\} \]

(b) \[ L^{-1} \left\{ \frac{1}{s \sinh(as)} \right\} \]

(c) \[ F^{-1} \left\{ \frac{\sinh(\alpha)}{\alpha \cosh(\alpha)} \right\} \]

Q3) Consider the following boundary value problem

\[ \frac{\partial u}{\partial t} = (a + \frac{\partial}{\partial t}) \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, 0 < a, t \]

\[ u(0,t) = 1, \quad \lim_{x \to \infty} u(x,t) < \infty \]

\[ u(x,0) = u_{xx}(x,0) \]

Using the Laplace transform show that \[ U(x,s) = \frac{1}{s} \exp(-x\sqrt{s}) \frac{s}{s+a} \]

Show that branch cuts are associated with \( \sqrt{s}, \sqrt{s+a} \) along negative real axis of \( s \)-plane.

Use Bromwich contour to show

\[ u(x,t) = 1 - \frac{2}{\pi} \int_{0}^{\infty} \exp(-a \eta^2) \frac{\sin(x \eta)}{1 + \eta^2 \eta(1 + \eta^2)} \, d\eta \]

Finally show that \( u(x,0) = e^{-x} \).

Q4) Solve using the Fourier transforms

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - u = 0, \quad -\infty < x < \infty, 0 < x < 1, \]

\[ \frac{\partial u(x,0)}{\partial z} = 0, \quad u(x,1) = e^{-x} H(x), \quad -\infty < x < \infty. \]