Prob. 1
If $E_1, E_2, \ldots$ is a collection of measurable subsets of $E$ with $\sum_{k=1}^{\infty} m(E_k) < \infty$, then the set of points that belong to infinitely many sets $E_k$ has measure zero.

Prob. 2
Suppose $f$ is integrable on $[0, b]$ and $g(x) = \int_x^b f(t) \frac{dt}{t}$, for $0 < x \leq b$. Prove that $g$ is integrable over $[0, b]$ and $\int_0^b g(x) \, dx = \int_0^b f(t) \, dt$.

Prob. 3
Suppose $F$ is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x)$ denote the distance from $x$ to $F$, that is

$$\delta(x) := d(x, F) = \inf \{|x - y| : y \in F\}.$$  

Consider $I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} \, dy$. Prove that

(a) $\delta$ is continuous, (b) $I(x) = \infty$ for each $x \notin F$, (c) $I(x) < \infty$ for $x \in F$

(Hint: investigate $\int_F I(x) \, dx$)

Prob. 4
Show that

(a) there exists a positive continuous function $f$ on $\mathbb{R}$ such that $f$ is integrable on $\mathbb{R}$ but yet $\limsup_{x \to \infty} f(x) = \infty$.

(b) However, if we assume that $f$ is uniformly continuous on $\mathbb{R}$ and integrable on $\mathbb{R}$, then $\lim_{|x| \to \infty} f(x) = 0$.

(Hint: For (a) construct a continuous version of the function $f(x) = n$ on $[n, n + \frac{1}{n^3})$, $n \geq 1$.)
Prob. 5
Consider the function \( f(x) = x^2 \sin(1/x^2), \ x \neq 0 \) with \( f(0) = 0 \). Show that \( f'(x) \) exists for every \( x \), but \( f' \) is not integrable on \([-1, 1]\).

Prob. 6
Prove that if \( f \) is of bounded variation on \([a, b]\) and if the function \( V(x) = V[a, x] \) is absolutely continuous on \([a, b]\), then \( f \) is absolutely continuous on \([a, b]\).

Prob. 7
Let \( f \) be real-valued and measurable on \( E \). Let \( 1 \leq p \leq \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Prove that
\[
\|f\|_p = \sup \int_E f g
\]
where the supremum is taken over all real-valued \( g \) such that \( \|g\|_{p'} \leq 1 \) and \( \int_E f g \) exists.

Prob. 8
Let \( \mu \) be a finite measure and \( \{f_n\} \) a sequence of integrable functions and \( f \) also an integrable function such that \( f_n \to f \) in measure. Suppose that
\[
\lim_{t \to \infty} \int_E \sqrt{1 + f^2_n} d\mu = \int_E \sqrt{1 + f^2} d\mu.
\]
Prove that \( f_n \to f \) in \( L^1(E) \).