1. Let $X$ and $Y$ be two Banach spaces and let $A \in L(X, Y)$. Suppose that for each $y \in Y$ there exists $x \in X$ such that

$$
\|x\| \leq \alpha \|y\|, \quad \|Ax - y\| \leq \beta \|y\|, \quad \alpha, \beta \in \mathbb{R}, \ \beta < 1.
$$

Prove that the equation $Au = f$ has for each $f \in Y$ a solution $u \in X$ such that

$$
\|u\| \leq \frac{\alpha}{1 - \beta} \|f\|.
$$

2. Consider $A : C([0, 1] : \mathbb{R}) \to C([0, 1] : \mathbb{R})$ defined by

$$(Ax)(t) = x(0) + tx(1).$$

Find the spectrum $\sigma(A)$, the spectral radius $r_{\sigma}(A)$ and the resolvent operator $R_{\lambda}(A) = (\lambda I - A)^{-1}$.

3. Let $H$ be a Hilbert space and let $A \in L(H)$ be compact and symmetric. Denote by $\overline{B}_1(0)$ the unit ball in $H$, centered at zero.

(a) Show that $F : \overline{B}_1(0) \to \mathbb{R}$, defined by $F(x) = (Ax, x)$ attains a maximum and a minimum. (Hint. Use Banach-Alaoglu Theorem).

(b) Let $\lambda(A) = \sup\{F(x) : x \in \overline{B}_1(0)\}$ and $\mu(A) = \inf\{F(x) : x \in \overline{B}_1(0)\}$.

Show that

(i) if $\lambda(A) > 0$ and $(Au, u) = \lambda(A)$ with $\|u\| = 1$ then $u$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda(A)$,

(ii) if $\mu(A) < 0$ and $(Av, v) = \mu(A)$ with $\|v\| = 1$ then $v$ is an eigenvector of $A$ corresponding to the eigenvalue $\mu(A)$.

(c) Let $H = L^2(0, 1; \mathbb{R})$. Consider $A : H \to H$ defined by

$$(Au)(x) = x \int_0^1 yu(y)dy.$$ 

Show that $A$ is symmetric, compact and find $\lambda(A)$, $\mu(A)$.

4. Assume that $a \in C^\infty(\mathbb{R})$ and $f$ is a generalized function. Show that

(i) $(af)' = af' + a'f$

(ii) $a(x)\delta'(x) = a(0)\delta'(0) - a'(0)\delta(x)$

(iii) $x\delta' = -\delta$ and $x^2\delta' = 0$. 


5. Consider the differential equation

\[ Lu(x) = a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x), \]

with continuous coefficients on the interval \([a, b]\), \(a_2(x) \neq 0\), and \(f\) is a piecewise continuous function on \([a, b]\).

A solution \(u\) is required to satisfy the two boundary conditions

\[
\begin{align*}
B_1 u & : = \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b) = \gamma_1, \\
B_2 u & : = \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2.
\end{align*}
\]

(i) Find the formal adjoint operator \(L^*\) and the necessary condition to have \(L^* = L\).

(ii) Show that the necessary and sufficient condition for self-adjointness is

\[
a_2(a) \det \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = a_2(b) \det \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.
\]

6. Solve the boundary value problem

\[
\begin{cases}
-u''(x) = f(x), & -1 < x < 1, \\
B_1 u = \int_{-1}^{1} x u(x) \, dx = \gamma_1 \\
B_2 u = \int_{-1}^{1} u(x) \, dx = \gamma_2
\end{cases}
\]