Math 102 Quiz 5 Date 23/07/2011

Ex1: Use trigonometric Substitution to evaluate

a) $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$, b) $\int x\sqrt{1-x^4}\,dx$

Ex2 Use partial fraction decomposition to evalute

a) $\int \frac{1}{x^3-1}\,dx$, b) $\int \frac{dx}{x^4-x^2}$

Solution:

Ex1: a) $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=\frac{3}{4}\left[\frac{(x+\frac{1}{2})^2}{\frac{3}{4}}+1\right]$

$=\frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]$. Thus

$I=\int \frac{x}{\sqrt{x^2+x+1}}\,dx=\frac{2}{\sqrt{3}}\int \frac{x}{\sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}}\,dx$. Let $\frac{2x+1}{\sqrt{3}}=\tan\theta$

with $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. So $x=\frac{1}{2}\left[\sqrt{3}\tan\theta-1\right]$

and $dx=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$. Hence:

$I=\frac{2}{\sqrt{3}}\int \frac{\sqrt{3}}{2}\,\frac{\frac{1}{2}\left[\sqrt{3}\tan\theta-1\right]}{\sqrt{\tan^2\theta+1}}\sec^2\theta\,d\theta=\int \frac{1}{2}\left(\sqrt{3}\tan\theta-1\right)\sec\theta\,d\theta$

$=\frac{\sqrt{3}}{2}\int \tan\theta\sec\theta\,d\theta-\frac{1}{2}\int\sec\theta\,d\theta$

$=\frac{\sqrt{3}}{2}\sec\theta-\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|+C$

$=\frac{\sqrt{3}}{2}\sec\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]-\frac{1}{2}\ln\left|\sec\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]+\frac{2x+1}{\sqrt{3}}\right|+C$

b) Let $u = x^2$, $du = 2x\,dx$. Thus $\int x\sqrt{1-x^4}\,dx = \int \frac{1}{2}\sqrt{1-u^2}\,du$

Let $u = \sin\theta$, with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then

$$\frac{1}{2}\int \sqrt{1-u^2}\,du = \frac{1}{2}\int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta = \frac{1}{2}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

Ex2. a) $x^3 - 1 = (x-1)(x^2+x+1)$. So $\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

• To find A, we multiply each side by $(x-1)$ and we take $x = 1$:

$$\left.\frac{1}{x^2+x+1}\right|_{x=1} = A + (x-1)\left.\frac{Bx+C}{x^2+x+1}\right|_{x=1} \quad \text{i.e} \quad \boxed{A = \frac{1}{3}}$$

• $$\frac{1}{x^3-1} = \frac{1}{3(x-1)} + \frac{Bx+C}{x^2+x+1} \Leftrightarrow \frac{3}{x^3-1} = \frac{x^2+x+1+3(x-1)(Bx+C)}{x^3-1}$$

$$= \frac{x^2+x+1+3Bx^2+3Cx-3Bx-3C}{x^3-1} = \frac{(3B+1)x^2+(1+3C-3B)x+(1-3C)}{x^3-1}$$

By identification, we have: $1-3C = 3$ i.e $C = -\frac{2}{3}$
and $3B+1 = 0$ i.e $B = -\frac{1}{3}$. Hence

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x+4}{x^2+x+1}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\int\left(\frac{2x+1}{x^2+x+1} + \frac{3}{x^2+x+1}\right)dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{2}\int \frac{dx}{x^2+x+1}.$$

Since $\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{2}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

Hence $\int \frac{dx}{x^3-1} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

Date No.

b) $\int \frac{dx}{x^4 - x^2} = \int \frac{dx}{x^2(x^2-1)} = \int \frac{dx}{x^2(x-1)(x+1)}$

$$\frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

- To find C, we multiply both sides by $(x-1)$ and we take $x=1$, we find that $C = \frac{1}{2}$.
- To find D, we multiply both sides by $(x+1)$ and we take $x=-1$, we find: $D = -\frac{1}{2}$.
- To find B, we multiply both sides by x^2 and we take $x=0$, we find: $B = -1$
- To find A, we multiply both sides by x and we take the limit when x goes to $+\infty$: $\lim\limits_{x \to +\infty} \frac{1}{x(x-1)(x+1)} = A + \lim\limits_{x\to+\infty}\left(\frac{B}{x} + \frac{Cx}{x-1} + \frac{Dx}{x+1}\right)$

Thus $0 = A + C + D$, hence $A = -C - D = 0$

Therefore $\frac{1}{x^2(x-1)(x+1)} = -\frac{1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$. Thus

$$\int \frac{dx}{x^4 - x^2} = \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$