Name: ____________________________ ID#: ______________
Instructor: ______________________ Sec #: ______ Serial #: ______

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

<table>
<thead>
<tr>
<th>Question #</th>
<th>Marks</th>
<th>Maximum Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>105</td>
</tr>
</tbody>
</table>
Q:1 (15 points) Use separation of variables method to find the nontrivial solution of the heat equation

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 2\pi, \quad t > 0 \]

subject to the boundary and initial conditions

\[
\begin{align*}
u(0, t) &= 0, \quad u(2\pi, t) = 0, \quad t > 0 \\
u(x, 0) &= x, \quad 0 < x < 2\pi.
\end{align*}
\]
Q.2: (15 points) Solve the wave equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions

$u(0, t) = 0$, $u(\pi, t) = 0$, $u(x, 0) = \sin(2x)$ and $u(x, y)$ is bounded as $y \to \infty$.

(Use separation of variables constant $\lambda = \alpha^2$)
Q:3 (15 points) Use separation of variables method to find the nontrivial solution of the wave equation

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 3, \quad t > 0 \]

subject to the boundary conditions

\[
\begin{align*}
    u(3, t) &= 0, \quad t > 0 \\
    u(r, 0) &= 0, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 1, \quad 0 < r < 3
\end{align*}
\]

solution is bounded at \( r = 0 \).
Q:4 (15 points) Find the steady-state temperature $u(r, \theta)$ in a sphere of radius 1 by solving the problem
\[
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \quad 0 < r < 1, \ 0 < \theta < \pi,
\]
subject to the boundary condition
\[
u(1, \theta) = 3 \cos(\theta), \quad 0 < \theta < \pi.
\]
Q:5 (15 points) Use Laplace transform to solve the problem

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0 \]

subject to the boundary and initial conditions

1. \( u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0 \)
2. \( u(x, 0) = 2 \sin(3\pi x) + 3 \sin(5\pi x), \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0, \quad 0 < x < 1. \)
Q:6 (15 points) If \( f(x) = \frac{1}{\pi} \int_0^\infty [A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)] \, d\alpha \) is the Fourier integral representation of
\[
\begin{cases}
0, & x < 0 \\
\sin x, & 0 \leq x \leq \pi \\
0, & x > \pi 
\end{cases}
\]
then show that \( A(\alpha) = \frac{1 + \cos(\alpha \pi)}{1 - \alpha^2} \), \( B(\alpha) = \frac{\sin(\alpha \pi)}{1 - \alpha^2} \)
and \( f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \alpha x + \cos \alpha (x-\pi)}{1 - \alpha^2} \, d\alpha \).
Q:7 (15 points) Use appropriate Fourier transform to find the temperature $u(x,t)$ in a semi infinite rod if $u(0,t) = 10, \ t \geq 0$ and $u(x,0) = 0, \ x > 0$. 