Math 132 Quiz 7 Date 29/10/2011

Let $f(x) = \dfrac{-3x^2+2x-5}{3x^2-2x-1}$

a) Find the intervals on which f is decreasing or increasing

b) Find any relative or absolute extremum

c) Find the intervals on which f is concave up or concave down

d) Find the inflection points if any

e) Find all the asymptotes of f.

h) Sketch the graph of f.

Solution:

a) $3x^2-2x-1=0$ iff $x=1$ or $x=-\frac{1}{3}$. Therefore the domain of f is $\mathbb{R}\setminus\{1,-\frac{1}{3}\}$.

$$f'(x) = \frac{(-6x+2)(3x^2-2x-1)-(6x-2)(-3x^2+2x-5)}{(3x^2-2x-1)^2}$$

$$= \frac{-18x^3+12x^2+6x+6x^2-4x-2+18x^3-12x^2+30x-6x^2+4x-10}{(3x^2-2x-1)^2}$$

$$= \frac{36x-12}{(3x^2-2x-1)^2} = 3\,\frac{12x-4}{(3x^2-2x-1)^2}$$ So $f'(x)=0$ iff $x=\frac{1}{3}$

x	$-\infty$		$-1/3$		$1/3$		1		$+\infty$
$f'(x)$		$-$	‖	$-$	0	$+$	‖	$+$	
$f(x)$	-1	↘ $-\infty$	‖	$+\infty$ ↘	$f(1/3)$	↗ $+\infty$	‖	$-\infty$ ↗	-1

f is increasing on $\left(\frac{1}{3}, 1\right) \cup (1, +\infty)$

f is decreasing on $(-\infty, -1/3) \cup \left(-\frac{1}{3}, \frac{1}{3}\right)$

b) $\frac{1}{3}$ is a relative minimum.

c) $$f''(x) = 12\left[\frac{3(3x^2-2x-1)^2 - 2(6x-2)(3x-1)(3x^2-2x-1)}{(3x^2-2x-1)^4}\right]$$

$$= 12 \; \frac{9x^2-6x-3-36x^2+12x+6x-2}{(3x^2-2x-1)^3}$$

$$= 12 . \frac{-27x^2+12x-5}{(3x^2-2x-1)^3}.$$

The discriminant of the numerator is negative. Thus the numerator is never zero (has no roots) and it is always negative (the sign of its leading coefficient). Therefore:

x	$-\infty$	$-1/3$		1		$+\infty$
$f''(x)$	$-$	‖	$+$	‖	$-$	

f is concave up on $(-1/3, 1)$

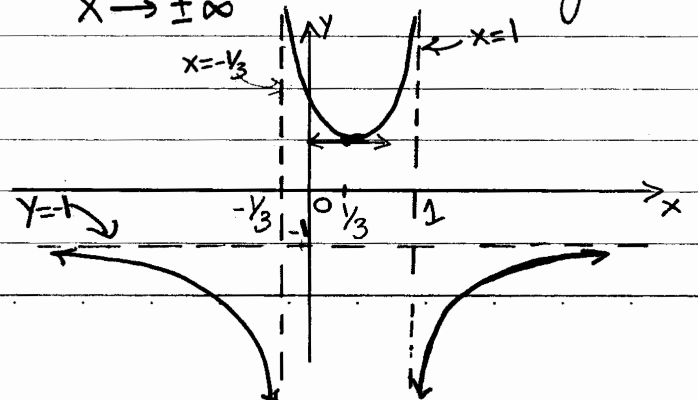
f is concave down on $(-\infty, -1/3)$ and $(1, +\infty)$

d) f has no inflection point.

e) •) $x = -\frac{1}{3}$, $x = 1$ are the vertical asymptotes

•) Since $\lim_{x \to \pm\infty} f(x) = -1$, $y = -1$ is a horizontal asymptote.

g)



Math 132 **Quiz 8** Date 14/11/2011

Ex1: Use differentials to approximate $e^{0.001}$

Ex2 Determine the indefinite integral $\int\left(\frac{2}{\sqrt[5]{x}} - \frac{1}{x}\right)dx$

Ex3 Find y subject to: $y'' = -3x^2 + 4x$; $y'(1) = 2$; $y(1) = 3$.

Solution

Ex1: Let $f(x) = e^x$. We have that $0.001 = \overbrace{0}^{x} + \overbrace{0.001}^{dx}$

So $f(0 + 0.001) \approx f(0) + f'(0)(0.001)$

Since $f'(x) = e^x$, $f(0) = f'(0) = 1$. Therefore

$f(0.001) \approx 1 + 0.001$ i.e $e^{0.001} \approx 1.001$

Ex2 $\int\left(\frac{2}{\sqrt[5]{x}} - \frac{1}{x}\right)dx = 2\int x^{-1/5}\,dx - \int \frac{1}{x}dx$

$= 2\,\frac{x^{-1/5+1}}{-\frac{1}{5}+1} - \ln|x| + C = \frac{5}{2}x^{\frac{4}{5}} - \ln|x| + C$

Ex3: $y' = -x^3 + 2x^2 + C$. Since $y'(1) = 2$, we have that

$-1 + 2 + C = 2$ i.e $C = 1$. Thus $y' = -x^3 + 2x^2 + 1$

Now $y = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + x + C$. Since $y(1) = 3$, we have that:

$-\frac{1}{4} + \frac{2}{3} + 1 + C = 3$ i.e $C = \frac{19}{12}$

Hence $y = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + x + \frac{19}{12}$

Math 132 Quiz 9 Date 21/11/2011

Determine the following indefinite integrals

a) $I = \int \frac{xe^{x^2}}{\sqrt{e^{x^2}+2}}\,dx$, b) $J = \int \frac{5}{(3x+1)\left[1+\ln(3x+1)\right]^2}\,dx$

c) $K = \int \sqrt{x}\,\sqrt{(8x)^{3/2}+3}\,dx$

Solution

a) Let $u = e^{x^2}+2$ Then $du = 2xe^{x^2}dx$. Thus $xe^{x^2}dx = \frac{1}{2}du$

Therefore $I = \int \frac{1}{2\sqrt{u}}\,du = \sqrt{u} + c = \sqrt{e^{x^2}+2} + c$

b) Let $u = 1+\ln(3x+5)$ Then $du = \frac{3}{3x+5}dx$ Thus

$J = \frac{5}{3}\int \frac{1}{u^2}\,du = -\frac{5}{3u} + c = -\frac{5}{3\left[1+\ln(3x+1)\right]} + c$

c) Let $u = (8x)^{3/2}+3$. Then $du = \frac{3}{2}\cdot 8\cdot (8x)^{1/2}dx = 24\sqrt{2}\sqrt{x}\,dx$

Therefore $\sqrt{x}\,dx = \frac{1}{24\sqrt{2}}\,du$ Hence

$K = \int \frac{1}{24\sqrt{2}}\sqrt{u}\,du = \frac{2}{3}\,\frac{1}{24\sqrt{2}}\,u^{3/2} + c$

$= \frac{1}{36\sqrt{2}}\,u^{3/2} + c = \frac{1}{36\sqrt{2}}\left[(8x)^{3/2}+3\right]^{3/2} + c$

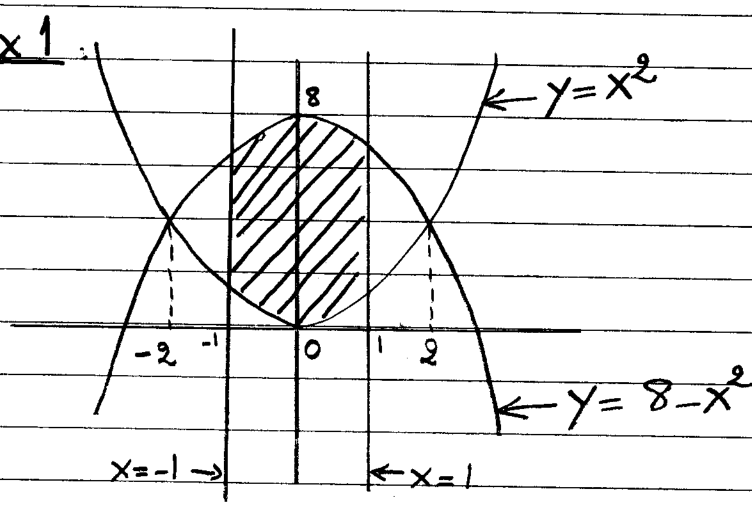
Math 132 Quiz 10 Date 02/12/2011

Ex1 Sketch the graph, then evaluate the area of the region bounded by $y = 8 - x^2$, $y = x^2$, $x = -1$ and $x = 1$

Ex Find $\int (2^x + x)^2 \, dx$

Solution

Ex 1



$$A = \int_{-1}^{1} \left[(8 - x^2) - x^2 \right] dx$$

$$= \int_{-1}^{1} (8 - 2x^2) \, dx = 2 \int_0^1 (8 - 2x^2) \, dx$$

$$= 2 \left[8x - \frac{2}{3} x^3 \right]_0^1$$

$$= 4 \left(4 - \frac{1}{3} \right) = \frac{44}{3}$$

Ex2 $I = \int \left(2^{2x} + 2x2^x + x^2 \right) dx = \int 2^{2x} dx + 2 \int x 2^x dx + \int x^2 dx$

We have that: $\cdot) \int 2^{2x} dx = \frac{1}{2 \ln 2} 2^{2x} + C$; $\cdot) \int x^2 dx = \frac{x^3}{3} + C$.

$\cdot)$ Let $u(x) = x$, $u'(x) = 1$ & $v'(x) = 2^x$, $v(x) = \frac{2^x}{\ln 2}$ Then

$$\int x 2^x dx = \frac{x 2^x}{\ln 2} - \frac{1}{\ln 2} \int 2^x dx = \frac{x 2^x}{\ln 2} - \frac{1}{(\ln 2)^2} 2^x + C$$

Hence:

$$I = \frac{1}{2 \ln 2} 2^{2x} + \frac{2.2^x}{\ln 2} \left(x - \frac{1}{\ln 2} \right) + \frac{x^3}{3} + C$$

Math 132 Quiz 11 Date 13/12/2011

a) $I = \int \cos x \cos(2x)\,dx$

1st method: Since $\cos(2x) = 2\cos^2 x - 1$, we have

$$I = \int \cos x\,(2\cos^2 x - 1)\,dx = \int (2\cos^3 x - \cos x)\,dx$$

$$= 2\int \cos^3 x\,dx - \underbrace{\int \cos x\,dx}_{=\ \sin x + C}$$

Now: $\int \cos^3 x\,dx = \int \cos x\,(1 - \sin^2 x)\,dx = \int (\cos x - \cos x \sin^2 x)\,dx$

$$= \sin x - \frac{1}{3}\sin^3 x\,dx$$ Therefore:

$$I = 2\sin x - \frac{2}{3}\sin^3 x - \sin x + C = \sin x - \frac{2}{3}\sin^3 x + C$$

2nd method: $\cos(2x) = \cos^2 x - \sin^2 x$. So:

$$I = \int (\cos^3 x - \cos x \sin^2 x)\,dx = \int \cos^3 x\,dx - \int \cos x \sin^2 x\,dx$$

$$= \sin x - \frac{1}{3}\sin^3 x - \frac{1}{3}\sin^3 x + C = \sin x - \frac{2}{3}\sin^3 x + C$$

b) $I = \int x \tan^2 x\,dx$ Let $u(x) = x$, $u'(x) = 1$

$v'(x) = \tan^2 x$, $v(x) = \tan x - x$

$$I = x(\tan x - x) - \int (\tan x - x)\,dx$$

$$= x(\tan x - x) - \int \tan x\,dx + \int x\,dx$$

$$= x(\tan x - x) + \ln|\cos x| + \frac{x^2}{2} + C$$

$$= x\tan x + \ln|\cos x| - \frac{x^2}{2} + C$$

c) $I = \int \sin(2x) \cos^5 x\,dx = \int 2 \sin x \cos x \cos^5 x\,dx$

$= 2\int \sin x \cos^6 x\,dx$ Let $u = \cos x$, $du = -\sin x\,dx$

$I = -2\int u^6\,dx = -\frac{2}{7} u^7 + C = -\frac{2}{7} \cos^7 x + C$

Math 132 Quiz 12 Date 17/12/2011

Let $f(x,y) = (y^2-4)(e^x-1)$

a) Find $f_x(x,y)$ and $f_y(x,y)$

b) Find all critical points of f

c) Find $f_{xx}(x,y)$, $f_{yy}(x,y)$ and $f_{xy}(x,y)$

d) Determine the nature of each critical point.

<u>Solution</u>

a) $f_x(x,y) = (y^2-4)e^x$; $f_y(x,y) = 2y(e^x-1)$

b) Setting $f_x(x,y) = 0 = f_y(x,y)$ implies

$$\begin{cases} (y^2-4)e^x = 0 \\ \& \\ 2y(e^x-1) = 0 \end{cases} \quad \underline{i.e} \quad \begin{cases} y = \pm 2, \ (e^x > 0 \text{ for all } x) \\ \& \\ y = 0 \text{ or } x = 0 \end{cases}$$

Hence f has two critical points $(0,2)$ and $(0,-2)$

c) $f_{xx}(x,y) = (y^2-4)e^x$; $f_{yy}(x,y) = 2(e^x-1)$

$f_{xy}(x,y) = 2ye^x$

d) •) $D(0,2) = 0 - (4)^2 = -16 < 0$. Hence $(0,2)$ is a saddle point.

•) $D(0,-2) = 0 - (-4)^2 = -16 < 0$ Hence $(0,-2)$ is a saddle point.