Q1. (i) Find an equation of the plane passing through the points (1,2,3), (-1,2,0) and perpendicular to the plane \( x + 2y + 3z = 1 \). (7 pts)

- A vector normal to the plane \( x + 2y + 3z = 1 \) is \( \mathbf{n}_1 = \langle 1, 2, 3 \rangle \).
- A vector passing through the points (1,2,3), (-1,2,0) is \( \mathbf{v} = \langle 2, 0, 3 \rangle \).
- A vector \( \mathbf{n} \) normal to the required plane can be found by \( \mathbf{n} = \mathbf{n}_1 \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} = 6\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \).

Using the point (1,2,3) and the normal \( \mathbf{n} = \langle 6, 3, -4 \rangle \), we find the equation of the required plane to be:

\[ 6(x-1) + 3(y-2) - 4(z-3) = 0 \]

\[ \Rightarrow 6x + 3y - 4z = 0 \]

(ii) Find the distance between the planes: \( x - 2y + 3z = 1 \) and \( -2x + 4y - 6z = 1 \). (5 pts)

- We choose the point (1,0,0) on the first plane.
- We write the equation of the 2nd plane in the general form as \( -2x + 4y - 6z - 1 = 0 \).
- We use the formula to find the distance

\[
D = \frac{|-2(1) + 4(0) - 6(0) - 1|}{\sqrt{4 + 16 + 36}} = \frac{3}{\sqrt{56}}
\]
Q2. Let $f(x, y) = \ln(\sqrt{16 - 4x^2 - y^2})$.

(i) Find and sketch the domain of $f$.

$$D = \{(x, y) : 16 - 4x^2 - y^2 > 0\}$$

$$= \{(x, y) : 4x^2 + y^2 < 16\} = \{(x, y) : \frac{x^2}{4} + \frac{y^2}{16} < 1\}$$

This is the interior of the ellipse sketched below.

(ii) Find the range of $f$.

$$0 < \sqrt{16 - 4x^2 - y^2} \leqslant 4 \Rightarrow 0 < \ln(\sqrt{16 - 4x^2 - y^2}) \leqslant \ln 4$$

$$\Rightarrow f(x) = \ln(\sqrt{16 - 4x^2 - y^2}) \text{ has the range }$$

$$R = (-\infty, \ln 4]$$

(iii) Write an equation of the level curve of $f$ which passes through the point $(1,1)$.

$$f(1,1) = \ln(\sqrt{11})$$

$$\Rightarrow \text{The level curve that passes through } (1,1) \text{ has an equation }$$

$$\ln(\sqrt{16 - 4x^2 - y^2}) = \ln(\sqrt{11})$$

$$\Rightarrow 16 - 4x^2 - y^2 = 11 \Rightarrow 4x^2 + y^2 = 5.$$
Q3. Find parametric equations of the normal line to the surface $\ln \left( \frac{x}{y-z} \right) = x - 1$ at the point $(1, 4, 3)$.

Let $F(x, y, z) = \ln \left( \frac{x}{y-z} \right) - x + 1$.

$\Rightarrow$ A normal vector to the given surface at the given point is

$\vec{n} = \nabla F(1, 4, 3) = \left\langle \frac{1}{x}, -1, \frac{1}{y-z} \right\rangle |_{(1, 4, 3)}$

$= \left\langle 0, -1, 1 \right\rangle$

$\Rightarrow$ The parametric equations of the normal line at $(1, 4, 3)$ are

$x = 1 \quad y = 4 - t \quad z = 3 + t$
Q4. The values of \( z = f(x, y) \) and its partial derivatives at \((2, -2)\) are given in the following table:

<table>
<thead>
<tr>
<th>( f(2, -2) )</th>
<th>( f_x (2, -2) )</th>
<th>( f_y (2, -2) )</th>
<th>( f_{xx} (2, -2) )</th>
<th>( f_{xy} (2, -2) )</th>
<th>( f_{yy} (2, -2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-5</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>-3</td>
</tr>
</tbody>
</table>

If \( x = r^2 + s^2 \) and \( y = 2rs \), then find

(i) \( \left. \frac{\partial z}{\partial s} \right|_{(r, s) = (1, -1)} \)

\[
\frac{\partial^2 z}{\partial s \partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2s \frac{\partial^2 z}{\partial x^2} + 2r \frac{\partial^2 z}{\partial y^2}
\]

When \( r = 1, s = -1 \) \( \Rightarrow x = 2, y = -2 \)

\[
\Rightarrow \left. \frac{\partial^2 z}{\partial s \partial r} \right|_{(r, s) = (1, -1)} = 2 (-1)(-5) + 2(1)(3) = 16
\]

(ii) \( \left. \frac{\partial^2 z}{\partial r \partial s} \right|_{(r, s) = (1, -1)} \)

\[
\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial}{\partial r} \left[ \frac{\partial z}{\partial s} \right] = \frac{\partial}{\partial r} \left[ 2s \frac{\partial^2 z}{\partial x^2} + 2r \frac{\partial^2 z}{\partial y^2} \right]
\]

\[
= 2s \frac{\partial}{\partial r} \left[ \frac{\partial^2 z}{\partial x^2} \right] + 2r \frac{\partial}{\partial r} \left[ \frac{\partial^2 z}{\partial y^2} \right]
\]

\[
= 2s \left[ 2r \frac{\partial^2 z}{\partial x^2} + 2s \frac{\partial^2 z}{\partial y^2} \right] + 2r \left[ 2r \frac{\partial^2 z}{\partial x \partial y} + 2s \frac{\partial^2 z}{\partial y^2} \right]
\]

\[
= 4sr \frac{\partial^2 z}{\partial x^2} + (4s^2 + 4r^2) \frac{\partial^2 z}{\partial x \partial y} + 4sr \frac{\partial^2 z}{\partial y^2}
\]

When \( r = 1, s = -1 \) \( \Rightarrow x = 2, y = -2 \)

\[
\left. \frac{\partial^2 z}{\partial r \partial s} \right|_{(r, s) = (1, -1)} = 4(-1)(1)(4) + (4(-1)^2 + 4(1)^2)(-3) + 4(-1)(1)(7) + 2(3)
\]

\[
= -16 - 24 - 28 + 6 = -62
\]
Q5. Find the absolute maximum and absolute minimum of \( f(x, y) = x(y^2 - 1) \) on the region 
\[ D = \{ (x, y) : x^2 + y^2 \leq 28 \} \].

We find the critical points inside \( D \):
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 0 \\
\frac{\partial f}{\partial y} &= y^2 - 1 = 0 \implies y = \pm 1 \\
2xy &= 0 \implies x = 0 \text{ or } y = 0
\end{align*}
\]
\( \implies \) The critical points are \((0, 1)\) and \((0, -1)\) with 
\[ f(0, \pm 1) = 0 \]

On the boundary \( x^2 + y^2 = 28 \) \( \implies f \) becomes
\[ f = x(28 - x^2 - 1) = 27x - x^3 \]
\( -\sqrt{28} \leq x \leq \sqrt{28} \)
\( \implies \frac{\partial f}{\partial x} = 27 - 3x^2 = 0 \implies x = \pm 3 \implies y = \pm \sqrt{19} \)
\[ f(3, \pm \sqrt{19}) = 54 \]
\[ f(-3, \pm \sqrt{19}) = -54 \]

and at the endpoints of the interval, we find
\[ x = \pm \sqrt{28} \implies y = 0 \implies f(\sqrt{28}, 0) = -28 \]
\[ f(-\sqrt{28}, 0) = 28 \]

Comparing all the above values obtained, we find that
\( f \) has max. value 54 at \((3, \sqrt{19})\) and \((3, -\sqrt{19})\)
and \( f \) has min. value -54 at \((-3, \sqrt{19})\) and \((-3, -\sqrt{19})\).