Math201-111 Exam 1
Solution

Part I [52 pts] (Written: Provide all necessary steps required in the solution.)

Q1. (i) Find \( \frac{d^2y}{dx^2} \) for the parametric curve \( C: \quad x = 3t^2 - t, \quad y = 2t + t^3. \) (5+5 pts)

Sol: \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 2}{6t - 1} \) and

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dx}{dt} = \frac{(6t)(6t - 1) - (3t^2 + 2)6}{(6t - 1)^3} = \frac{36t^2 - 6t - 18t^2 - 12}{(6t - 1)^3} = \frac{18t^2 - 6t - 12}{(6t - 1)^3}
\]

(ii) Find the interval(s) where \( C \) is concave up.

Sol: \( C \) is concave up if

\[
\frac{d^2y}{dx^2} = \frac{18t^2 - 6t - 12}{(6t - 1)^3} = \frac{6(3t + 2)(t - 1)}{(6t - 1)^2} > 0
\]

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The curve is concave up for \( x \in \left( -\frac{2}{3}, \frac{1}{6} \right) \cup (1, \infty) \)

Q2. Consider the vectors \( \vec{u} = -3\vec{i} + \vec{j} + 2\vec{k} \) and \( \vec{v} = \vec{i} + 2\vec{j} - 3\vec{k} \) (10 pts)

(i) Find the angle between \( \vec{u} \) and \( \vec{v} \).

Sol: \( \cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||} = \frac{-3 + 2 - 6}{\sqrt{14} \sqrt{14}} = \frac{-7}{14} = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}. \)

(ii) Find the projection of \( \vec{u} \) onto \( \vec{v} \)

Sol: \( \text{proj}_{\vec{v}} \vec{u} = \left( \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \right) \vec{v} = \frac{-7}{14}(1,2,-3) = \left( -\frac{1}{2}, -1, \frac{3}{2} \right). \)
Q3. Use the **scalar triple product** to determine whether the four points:  
\[ A(1,3,2), \ B(3,-1,6), \ C(5,2,0), \ D(3,6,-4) \]  
lie in the **same plane**.

**Sol:**  
\[ \vec{AB} = (2,-4,4), \ \vec{AC} = (4,-1,-2), \ \text{and} \ \vec{AD} = (2,3,-6) \]

\[
\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2(6+6) + 4(-24+4) + 4(12+2) = 24 - 80 + 56 = 0.
\]

Q4. Find the exact **length** of the **polar curve**:  
\[ r = \theta^2, \quad 0 \leq \theta \leq \pi / 4. \]  
(10 pts)

**Sol:**
\[
s = \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2} \ d\theta = \int_0^\pi \theta \sqrt{\theta^2 + 4} \ d\theta = \frac{1}{3} (\theta^2 + 4)^{3/2} \bigg|_0^\pi = \frac{1}{3} (\pi^2 + 4)^{3/2} - \frac{8}{3}.
\]

Q5. Consider the **polar curve** \( C: r = 2 + 4 \sin \theta \)  
(3 + 2 + 2 +5 pts)

(a) Show that \( C \) is **symmetric** about the vertical line \( \theta = \frac{\pi}{2} \).  
**Sol:**  
\[
r(\pi - \theta) = 2 + 4\sin(\pi - \theta) = 2 + 4(\sin \pi \cos \theta - \cos \pi \sin \theta) = 2 + 4\sin \theta = r(\theta)
\]

(b) Find the **polar coordinates** of the points where \( C \) intersects the **polar axis**.

**Sol:** \( C \) intersect polar axis at \( \theta = 0, r = 2 \), that is (2,0).  
\( C \) also intersect polar axis at pole, that is \( r=0 \).

\[
r = 2 + 4\sin \theta = 0 \ \Rightarrow \ \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.
\]

\[
\left(0, \frac{7\pi}{6}\right), \left(0, \frac{11\pi}{6}\right).
\]

(c) Find the **polar coordinates** of the points where \( C \) intersects the lines \( \theta = \frac{\pi}{2} \) and \( \theta = \frac{\pi}{4} \)

At \( \theta = \frac{\pi}{2}, r = 6 \) and at \( \theta = \frac{\pi}{4}, r = 2 + 2\sqrt{2} \)

The points are \( \left(6, \frac{\pi}{2}\right) \) and \( \left(2 + 2\sqrt{2}, \frac{\pi}{4}\right) \).
(d) **Plot** the points obtained in (b)-(c) and make use of (a) to **sketch the graph** of $C$ in the following polar chart:  

*Indicate important values of $r$ and $\theta$ in the outer circle of the chart*
Q1. If the end points of a diameter of a sphere lie at $A(1,4,-2)$ and $B(-7,1,2)$ then an equation of the sphere is given by

(a*) $x^2 + y^2 + z^2 + 6x - 5y = 7$
(b) $x^2 + y^2 + z^2 - 8x - 4y = 10$
(c) $x^2 + y^2 - z^2 + 6x - 4y = 7$
(d) $x^2 + y^2 + z^2 + 7x - 10y = 20$
(e) $x^2 + y^2 + z^2 + 6x + 4y = 12$

Sol: Mid point is $\left(\frac{1-7}{2}, \frac{4+1}{2}, \frac{-2+2}{2}\right) = \left(-3,\frac{5}{2},0\right)$
and radius is $R = \sqrt{(1+3)^2 + \left(4 - \frac{5}{2}\right)^2 + (-2-0)^2} = \sqrt{16 + \frac{9}{4} + 4} = \frac{\sqrt{89}}{2}$
equation of the sphere is

$\left(x + 3\right)^2 + \left(y - \frac{5}{2}\right)^2 + z^2 = \frac{89}{4}$
$x^2 + y^2 + z^2 + 6x - 5y = \frac{89}{4} - 9 - \frac{25}{4} = \frac{89 - 36 - 25}{4} = \frac{28}{4} = 7$

Q2. Suppose that a 3-D vector $\vec{v}$ lies below the $xy$-plane and has the direction angles $\alpha, \beta, \gamma$ with $x, y$ and $z$ axes respectively. If $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{3}$, then the value of $\gamma$ is given by

(a*) $\frac{2\pi}{3}$
(b) $\left(\sqrt{2} \pi\right)/2$
(c) $-1/2$
(d) $5\pi/6$
(e) $-1/\sqrt{2}$

Sol: $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \implies \frac{1}{2} + \frac{1}{4} + \cos^2 \gamma = 1 \implies \cos^2 \gamma = 1 - \frac{3}{4} = \frac{1}{2}$
Since the vector $\vec{v}$ lies below $xy$-plane, therefore, $\cos \gamma = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \implies \gamma = \frac{2\pi}{3}$.

Q3. A value of $\alpha$ for which the vectors $\vec{u} = 3\hat{i} + \alpha \hat{k}$ and $\vec{v} = 2\alpha \hat{i} - \hat{j}$ have the same length is given by

(a*) $\sqrt{8}/3$
(b) $\sqrt{5}/3$
(c) $\sqrt{8}/5$
(d) $\sqrt{7}/3$
(e) $\sqrt{5}/8$

Sol: $3^2 + \alpha^2 = 4\alpha^2 + 1 \implies 3\alpha^2 = 8 \implies \alpha = \frac{\sqrt{8}}{3}$
Q4. The area of the triangle with the vertices \((a,0,0), (0,2a,0)\) and \((0,0,3a)\) is

(a*) \(7a^2 / 2\)  
(b) \(5a^2 / 2\)  
(c) \(6a^3\)  
(d) \(7a\)  
(e) \(3a^3 / 2\)

**Sol:** Let \(\vec{u} = (-a, 2a, 0)\) and \(\vec{v} = (-a, 0, 3a)\) be two adjacent vectors. Then

\[
\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & 2a & 0 \\ -a & 0 & 3a \end{vmatrix} = 6a^2 \hat{i} + 3a^2 \hat{j} + 2a^2 \hat{k}
\]

and area \(A = \frac{1}{2} |\vec{u} \times \vec{v}| = \frac{1}{2} \sqrt{36a^4 + 9a^4 + 4a^4} = \frac{1}{2} \sqrt{49a^4} = \frac{7a^2}{2}\)

Q5. The area of the region inside the curve \(r = 3\sin \theta\) and outside the curve \(r = 2 - \sin \theta\) is

(a*) \(\int_{\pi/6}^{5\pi/6} (4\sin^2 \theta + 2\sin \theta - 2) d\theta\)  
(b) \(\int_{-\pi/6}^{\pi/6} (2\sin^2 \theta - 2\sin \theta - 1) d\theta\)  
(c) \(\int_{-\pi/6}^{\pi/6} (4\sin^2 \theta + \sin \theta + 3) d\theta\)  
(d) \(\int_{-\pi/6}^{\pi/6} (4\sin^2 \theta + 5\sin \theta - 2) d\theta\)  
(e) \(\int_{-\pi/6}^{\pi/6} (4\sin^2 \theta + 5\sin \theta - 2) d\theta\)

**Sol:** \(A = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9\sin^2 \theta - (2 - \sin \theta)^2) d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9\sin^2 \theta - 4 + 4 \sin \theta - \sin^2 \theta) d\theta\)

\[
= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (8\sin^2 \theta + 4 \sin \theta - 4) d\theta = \int_{\pi/6}^{5\pi/6} (4\sin^2 \theta + 2 \sin \theta - 2) d\theta
\]
Q6. The Cartesian equation of the curve $x = \ln t$, $y = \sqrt{t}$, $t \geq 1$ is given by

(a*) $y = e^{x^2}$, $x \geq 0$

(b) $y = e^x$, $x \geq 1$

(c) $y = e^{x^2}$, $x \geq 1$

(d) $y = e^x$, $x \geq 0$

(e) $y = e^{2x}$, $x \geq 0$

Sol: $x = \ln t$, $t \geq 0 \Rightarrow t = e^x, x \geq 0$

$y = \sqrt{t} = e^{x^2}, x \geq 0$.

Q7. The slope of the tangent line to the polar curve $r = \cos \theta + 1$ at $\theta = \frac{\pi}{2}$ is

(a*) 1

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) 0

(e) $-\frac{1}{2}$

Sol: $x = r \cos \theta = \cos \theta + \cos^2 \theta$, $\frac{dx}{d\theta} = -\sin \theta - 2 \sin \theta \cos \theta$

$y = r \sin \theta = \sin \theta + \sin \theta \cos \theta$, $\frac{dy}{d\theta} = \cos \theta + \cos 2 \theta$

\[
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta + \cos 2 \theta}{-\sin \theta - 2 \sin \theta \cos \theta}, \quad \text{and at } \theta = \frac{\pi}{2}, m = \frac{dy}{dx} = \frac{-1}{-1} = 1.
\]

Q8. Two forces $F$ and $G$ are acting on an object placed at the origin of the $xy$-plane with magnitudes 1 N and 2 N respectively.

If $F$ acts along the positive $y$-axis and $G$ makes an angle of $\theta = \frac{\pi}{3}$ with the positive $x$-axis, then the magnitude of the resultant force $F + G$ is

(a*) $\sqrt{5 + 2\sqrt{3}}$ N

(b) $\sqrt{1 + \sqrt{3}}$ N

(c) $\sqrt{2 + \sqrt{3}}$ N

(d) $5 + \frac{\sqrt{3}}{2}$ N

(e) $\sqrt{2 + 2\sqrt{3}}$ N

Sol: $\vec{F} = 0\hat{i} + 1\hat{j} = \hat{j}$ and $\vec{G} = 2 \cos \frac{\pi}{3} \hat{i} + 2 \sin \frac{\pi}{3} \hat{j} = \hat{i} + \sqrt{3} \hat{j}$

$\vec{F} + \vec{G} = \hat{i} + (1 + \sqrt{3})\hat{j}$ and $|\vec{F} + \vec{G}| = \sqrt{1 + (1 + \sqrt{3})^2} = \sqrt{1 + 1 + 2\sqrt{3} + 3} = \sqrt{5 + 2\sqrt{3}}$
KEY for 4 versions of Part II (MCQ's)

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