Q1) (14 Points) Define each of the following:

a. Numerically equivalent sets
b. Topologically equivalent spaces
c. A basis for a topological space
d. Dense subset
e. Frontier of a set
f. Hausdorff space
g. Normal space

Q2) (8 Points) Let $A$ and $B$ be two subsets of a set $X$ and let $f : X \rightarrow X$ be a function. Complete each of the following:

a. $f^{-1}(A \cap B)$ \hspace{2cm} $f^{-1}(A) \cap f^{-1}(B)$

b. $f^{-1}(A - B)$ \hspace{2cm} $f^{-1}(A) - f^{-1}(B)$

c. $f(f^{-1}(A))$ \hspace{2cm} $A$

d. $f^{-1}(f(A))$ \hspace{2cm} $A$
Q3) (12 Points) Consider the space \((\mathbb{Z}_+, \mathcal{I}_r)\). Let \(A = \{0,1\}\) and \(B = \{3n : n \in \mathbb{Z}_+\}\).

a. Is \(A\) \(\mathcal{I}_r\)-open?  
g. Is \(B\) \(\mathcal{I}_r\)-open?

b. Is \(A\) \(\mathcal{I}_r\)-closed?  
h. Is \(B\) \(\mathcal{I}_r\)-closed?

c. Find \(A^c\)  
i. Find \(B^c\)

d. Find \(\overline{A}\)  
j. Find \(\overline{B}\)

e. Find \(fr(A)\)  
k. Find \(fr(B)\)

f. Find \(A'\)  
l. Find \(B'\)

Q4) (8 Points) Let \(X\) be a set and consider the topologies \(\mathcal{I}_c\), \(\mathcal{I}_f\), and \(\mathcal{I}_d\) for \(X\).

a. How are \(\mathcal{I}_c\) and \(\mathcal{I}_f\) related, if at all?

b. How are \(\mathcal{I}_c\) and \(\mathcal{I}_d\) related, if at all?

c. If \(\mathcal{I}_c = \mathcal{I}_d\) what must be true about \(X\)?

d. If \(\mathcal{I}_c = \mathcal{I}_f\) what must be true about \(X\)?

Q5) (8 Points) True or false. Tick as true ( \(\checkmark\) ) or false ( \(\times\) ):

a. Any two countable sets are equivalent.

b. If \(A\) is countably infinite subset of an uncountable set \(B\), then \(B \sim B-A\)

c. Every subset of a topological space is either open or closed.

d. The set of all open rays is a basis for the usual topology for \(\mathbb{R}\).

e. The set of all open intervals is a subbasis for the usual topology for \(\mathbb{R}\).

f. The boundary of any set is closed.

g. The set \(\bigcup \left\{ [1/n, n] \mid n \in \mathbb{Z}_+ \right\}\) is closed set in \(\mathbb{R}\) with the usual topology.

h. Each boundary point of \(A\) is a limit point of a set \(A\).
Q6) (8 Points) Let \((X, \mathcal{I})\) be a topological space and let \(A \subseteq Y \subseteq X\).

a. Briefly describe how \(\mathcal{I}_{relY}\) is defined.

b. How are \(\overline{A}\) and \(\overline{A}_{relY}\) related?

c. How are \(A'\) and \(A_{relY}'\) related?

d. How are \(fr(A)\) and \(fr(A)_{relY}\) related?

Q7) (8 Points) Consider \(\mathbb{R}^2\) with the usual topology. Consider the subsets \(A = \{(x,0) \mid -2 \leq x \leq 2\}\) and \(B = \{(x,y) \mid x^2 + y^2 \geq 2\}\).

(a) Find \(A'\)

(b) Find \(B'\)

(c) Find \(fr(A)\)

(d) Find \(fr(B)\)

(e) Find \(A'\)

(f) Find \(B'\)

(g) Find \(\overline{A} \cap \overline{B}\)

Q8) (6 Points) Let \(X\) be uncountable set and consider the topologies \(\mathcal{I}_F, \mathcal{I}_C,\) and \(\mathcal{I}_D\) for \(X\). Circle the property which the space has.

a. \((X, \mathcal{I}_F)\) is \(\begin{array}{ccccccc} T_0 & T_1 & T_2 & \text{regular} & \text{normal} & T_3 & T_4 \end{array}\)

b. \((X, \mathcal{I}_C)\) is \(\begin{array}{ccccccc} T_0 & T_1 & T_2 & \text{regular} & \text{normal} & T_3 & T_4 \end{array}\)

c. \((X, \mathcal{I}_D)\) is \(\begin{array}{ccccccc} T_0 & T_1 & T_2 & \text{regular} & \text{normal} & T_3 & T_4 \end{array}\)
Q9) (10 Points) Let $(X, \mathcal{I})$ be a topological space and let $A \subseteq X$. Prove or disprove each of the following:

a. $X - \overline{A} = (X - A)^*$

b. $A^* = \overline{A}$

Q10) (10 Points) Let $X$ be a nonempty set and let $\{X_i\}$ be a class of topological spaces. Assume that we have a set of functions $\{f_i : X \to X_i\}$. The smallest topology on $X$ that will make each $f_i$ continuous is called the weak topology for $X$ generated by the $f_i$'s.

Assume that $X_i = \mathbb{R}$ with the usual topology $\mathcal{I}_u$ for each $i$, and consider the set $\{f_i : \mathbb{R} \to X_i\}$. Describe the weak topology generated by the $f_i$'s, where

a. $\{f_i\}$ is the set of all constant functions.

b. $\{f_i\}$ is the set of functions each of which is equal to $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Q11) (18 Points) Let $(X, \mathcal{I})$ be a topological space and let $A \subseteq X$. Prove each of the following:

a. The set $A$ is dense in $X$ if its complement has empty interior.

b. If the set $A$ has empty frontier, then it is both open and closed.