King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 535 [Functional Analysis I]

First Semester 2011 – 2012 (111)

Exam I: October 29, 2011 **Time: 2 hours**

1. (a) Define a convex set in a normed space *X*. Show that the closed unit ball of *X* is convex.

   (b) Define norm of a linear transformation from a normed space *X* into another normed space *Y*. Suppose that $f:\mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x) = x_1 + x_2 + x_3$ where $x = (x_1, x_2, x_3)$ and $\|x\| = \left(\sum_{i=1}^{3} |xi|^2\right)^{1/2}$. Show that $\|f\| = \sqrt{3}$.

2. (a) Let p be a fixed integer such that $1 \le p \le \infty$. Define $\ell_p = \left\{x = \{x_n\}: \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$. Consider ℓ_p under its usual sum norm $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ and prove that $\left(\ell_p, \|\cdot\|_p\right)$ is a Banach space.

   (b) Show that every finite dimensional subspace of a normed space *X* is closed in *X*.

3. (a) Let *X* and *Y* be normed spaces and $T: X \to Y$ a linear operator. Then prove that T is continuous if and only if it is bounded.

   (b) Describe meanings of the statement (do not provide the proof):
   Dual space of c_o under sup norm is ℓ_1.

4. (a) Let *E* be a normed space and *F* be a Banach space. Prove that the vector space *L (E, F)* of all linear and continuous transformations from *E* into *F* is a Banach space with respect to the norm.
   $$\|T\| = \sup_{0 \ne x \in E} \frac{\|Tx\|}{\|x\|}.$$

   (b) Let $f:[a,b] \to \mathbb{R}$ be a differentiable function such that there exist constants μ and γ satisfying $0 < \mu \le f'(x) \le \frac{1}{\gamma}$ and $f(a) < 0 < f(b)$. Use Banach contraction mapping principle to find a unique root $\bar{x} \in [a,b]$ of the equation $f(x) = 0$.