King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 535 [Functional Analysis I]
First Semester 2011-2012(111)

Exam II: December 21,2011 Time: 2 hours

1. (a) Let f be a bounded linear functional on a subspace Z of a normed space X. Prove that there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f on X and $\| f \|_Z = \| \tilde{f} \|_X$
(b) Define a norm on $\mathbb{R}^3$ by $\| (x, y, z) \| = | x | + | y | + | z |$
Let $S = \{(x, y, z) | x + 2y - z = 0\} \subseteq (\mathbb{R}^3, || \cdot ||)$.
Define a real linear functional f on S by $f(x, y, z) = x$. Find a non-trivial extension F of f on $\mathbb{R}^3$ with $\| F \| = 1$

2. (a) Let $\{T_n\}$ be a sequence of bounded linear transformations from a Banach space X into a normed space Y such as $\| T_n x \|$ is bounded for every $x \in X$. Then prove that the sequence $\{\| T_n \|\}$ is bounded.
(b) Let A be a subset of a normed space X. Use part (a) to show that A is bounded if and only if $f(A)$ is bounded for all $f \in X^*$.

3. (a)Let T be a bounded linear mapping of a Banach space E onto a Banach space F. If U is an open subset of E, then prove that $T(U)$ is an open subset of F.
(b) Give an example (alongwith necessary details) of a map from a normed space into a Banach space which is closed but not continuous.

4. (a) Let $(X, \| \cdot \|)$ be a real normed space. If the law of parallelogram holds in X, then prove that X is an inner product space.
(b) If x, y, z are in an inner product space, then show that
$\| z - x \|^2 + \| z - y \|^2 = \frac{1}{2} \| x - y \|^2 + 2 \| z - \frac{x + y}{2} \|^2$