1. (a) Every metric on a vector space is not necessarily a norm. Justify this statement by means of a suitable example.

(b) Define a contraction on a metric space. Let $T : [1, \infty) \to [1, \infty)$ be given by $Tx = \frac{25}{26} \left( x + \frac{1}{x} \right)$. Use Banach fixed point theorem to find a unique fixed point of $T$.

2. (a) Let $p$ be a fixed integer such that $1 \leq p \leq \infty$. Define $l_p = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$. Consider $l_p$ under its usual sum norm and show that the dual of $l_p$ is $l_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(b) Prove that a closed linear mapping of a Banach space $E$ into a Banach space $F$ is continuous.

3. (a) Let $\{\alpha_n\}$ be a sequence of reals. Define a sequence of functionals on $l_1$ (under its usual norm) by $f_n(x) = \sum_{k=1}^{n} \alpha_k \xi_k$, $x = \{f_k\} \in l_1$. Show that each $f_n$ is linear and continuous and $\|f_n\| = \max_{1 \leq k \leq n} |\alpha_k|$. Assume that $\sum_{k=1}^{\infty} \alpha_k \xi_k$ is convergent for every $\{\xi_k\} \in l_1$. Use uniform boundedness principle, to show that $\{\alpha_n\}$ is bounded.

(b) Let $x \neq 0$ be any element of a normed space $X$. Prove that there exists a bounded linear functional $f$ on $X$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Hence show that $\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|}$ where $X^*$ denotes dual of $X$.

4. (a) Let $f$ be a continuous linear functional on a Hilbert space $H$. Prove that there exists a unique $z \in H$ such that $f(x) = \langle x, z \rangle$ for all $x \in H$ and $\|f\| = \|z\|$.

(b) Let $M$ be a nonempty subset of a Hilbert space $H$. If the span of $M$ is dense in $H$, then show that $M^\perp = \{0\}$.

5. (a) Let $E$ be a normed space and suppose that $E^*$ is separable. Then prove that $E$ is separable.
(b) Let \( \{x_n\} \) be a sequence in a normed space \( X \). If \( \{x_n\} \) converges weakly to \( x \in X \), then show that:

(i) the sequence \( \{\|x_n\|\} \) is bounded.

(ii) for all \( f \in M^\ast \), we have \( f(x_n) \to f(x) \) where \( M^\ast \) is a strongly dense subset of \( X^\ast \).

6. (a) Prove that a bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.

(b) Let \( A \) be a complex Banach algebra with identity \( e \). Then prove that the set of all invertible elements of \( A \) is an open set. If \( x \in A \), then is it true that the spectrum of \( x, \sigma(x) \), is an empty set?