(1) Determine SVD of the matrix $A$ (show all work)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U = \quad \Sigma = \quad V =$$

(2) The singular value decomposition of $B$ is

$$B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .87 & .39 & -.30 \\ .30 & -.91 & -.30 \\ .39 & -.17 & .90 \end{bmatrix} \begin{bmatrix} 2.49 \\ 0 \\ 1.33 \end{bmatrix} = \begin{bmatrix} .23 & .97 \\ .97 & -.23 \end{bmatrix}$$

Express $B$ as the sum of rank-one matrices:

$$B = \begin{bmatrix} \end{bmatrix} + \begin{bmatrix} \end{bmatrix}$$
(3) The QR factorization of $A$ is:

$$
A = \begin{bmatrix}
1 & 3 \\
1 & 1 \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
0.771 & 0.5774 & 0.4082 \\
0.7071 & -0.5774 & -0.4082 \\
0 & 0.5774 & -0.8165
\end{bmatrix} \begin{bmatrix}
1.4142 & 2.8284 \\
0 & 1.7321 \\
0 & 0 & 0
\end{bmatrix}
$$

Solve the least square problem $Ax = b$ where $b = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

(4) Let $A \in R^{m \times n}, m > n$

a) Explain why the rank of $A$ is exactly the number of its nonzero singular values

b) If $A = U \Sigma V^T$ is a singular value decomposition, what are the singular value decomposition of $A^T, A^+$

[Note that $A^+ \in R^{m \times m}, A^+ = (A^T A)^{-1} A^T ]$

(5) Let $A \in R^{m \times n}, m > n$ be such that $\|Ax\|_2 \leq \|x\|_2$ for all $x \in R^n$. Show that $A$ has orthonormal columns; that is, that $A^T A = I_n$.

(6) Suppose the matrix $A \in R^{2m \times 2n}$ has the form

$$
A = \begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix}
$$

where $A_1, A_2 \in R^{m \times n}$ are nonzero matrices. Show that

$$
\frac{\|A\|_2}{\|A_1\|_F} \geq \frac{\|A_2\|_2}{\|A\|_F}
$$

[HINT: use SVD]

BONUS:

Given $A \in R^{m \times n}, m > n$ and a nonzero vector $v \in R^n$. The following algorithm overwrites $A$ with $HA$ where $P = I - 2vv^T / v^Tv$

```
function [A] = row.house(A, v)
    \beta = -2 / v^Tv
    \beta = \beta vA^T v
    A = A + \beta wv^T
```

Count how many flops are required for this algorithm. Write your answer in terms $n$ and $m$.

[A flop is a floating point operation. Each addition, subtraction, multiplication, division or square root counts as one flop.]