1) (a) Let $A \in \mathbb{R}^{m \times n}$. Use SVD to find the Schur decomposition of the symmetric matrices $A^T A$, $AA^T$, $\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$.

(b) Let $A \in C^{n \times n}$ and $\det(\lambda I - A) = c_0 + c_1 \lambda + \ldots + c_{n-1} \lambda^{n-1} + \lambda_n$. For given $x \in \mathbb{C}^n$ the matrix $K = [x, Ax, \ldots, A^{n-1} x] \in C^{n \times n}$ is called the Krylov matrix of $A$ at $x$. Suppose that $Q^T AQ = H$ is a transformation of $A$ to upper Hessenberg form with some orthogonal matrix $Q$. Let $q_1$ be the first column of $Q$ and from the Krylov matrix of $A$ at $x = q_1$. Show that $K = Q(e_1, He_1, \ldots, H^{n-1} e_1)$.

2) The boundary value problem

$$
\frac{d}{dx} \left[ a(x) \frac{dy}{dx} \right] + \lambda y = 0, \quad y(-1) = y(1) = 0
$$

is approximated by the eigenvalue problem

$$
- \frac{1}{h^2} \left[ a(x_k + \frac{h}{2})(z_{k+1} - z_k) - a(x_k - \frac{h}{2})(z_k - z_{k-1}) \right] = \lambda z_k, \quad h = 1, \ldots, n
$$

where $x_k = -1 + kh$, $h = 2/(n + 1)$, and $z_0 = z_{n+1} = 0$. Let $a(x) = 1 + x^2$. Use QR algorithm with shift to compute the smallest eigenvalue for $n=10, 20$.

3) Let $A \in \mathbb{R}^{m \times n}$, $m > n$.
   a) write a MATLAB function $[Q, L] = q_l(A)$ that compute the matrix $Q$ and $L$ (lower triangular) which you developed in problem 2 HW3.
   b) Apply the following Algorithm (which uses function $q_l$ in part a)

   ```matlab
   function [d]=ql_algoritm(A)
   n=size(A);
   for k=1:maxiter
       [Q,L]=ql(A);
       A=L*Q;
   end
   ```

   to the matrix

   $$
   B = \begin{bmatrix}
   2 & -1 & 0 & 0 \\
   -1 & 4 & -1 & 0 \\
   0 & -1 & 4 & -1 \\
   0 & 0 & -1 & 2
   \end{bmatrix}
   $$
4) Compute the eigenvalues of the matrix $A$ by using three different methods described below then fill the table.

METHOD-1: MATLAB command `eig`
METHOD-2: QR Algorithm (basic version – without shift)
METHOD-3: QR Algorithm with shift $\mu_k = h_{nn}^{(k)}$ (first find Hessenberg $H$)
METHOD-4: QR Algorithm with Wilkinson shift (first find Hessenberg $H$)

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of iteration in QR Algorithm</th>
<th>CPU-time (use tic-toc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>METHOD-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>METHOD-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>METHOD-3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>METHOD-4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$A = \begin{bmatrix} B & -I & 0 & 0 \\ -I & B & -I & 0 \\ 0 & -I & B & -I \\ 0 & 0 & -I & B \end{bmatrix}$$ where $B$ is defined in problem 3.