Problem 1.
i. Show that the kernel
\[ k(x,t) = \sum_{n=1}^{\infty} \frac{\sin(n+1)x \sin nt}{n^2}, \quad 0 \leq x, t \leq \pi \]
is a Hilbert-Schmidt kernel. Show it is not symmetric.

ii. Show that the corresponding integral operator \( K \) has no eigenvalues and compute \( \|K\| \).

iii. Show that for every \( \lambda \neq 0 \) the equation \( Ku - \lambda u = f \) has a unique solution.

iv. Show that \( \lambda = 0 \) is either in the continuous spectrum or in the residual spectrum of \( K \).

Problem 2.
Show that the eigenvalues of \( -(x^2u)' - \lambda u = 0, \quad 1 < x < e, \quad u(1) = u(e) = 0 \) are positive. Find the eigenvalues and normalized eigenfunctions.
(hint. Try solutions of the form \( u(x) = x^\mu \)).

Problem 3.
Consider the differential equation (\(*\)) \(-u'' + x^2u - \lambda u = 0, \quad -\infty < x < +\infty.\)
Both endpoints are singular.

i. Make the substitution \( u(x) = z(x) \exp(x^2/2) \). Show that \( z \) satisfies
\[ (**): -z'' - 2xz - z = \lambda z. \]

ii. Study the case \( \lambda = -1 \) and show that both endpoints are in the limit-point case.

iii. Construct the Green’s function and translate the problem into an integral equation. Show that the kernel is a symmetric Hilbert-Schmidt kernel with eigenfunctions forming an orthonormal basis.

iv. The Hermite polynomials are defined by
\[ H_n(x) = (-1)^n \exp x^2 \frac{d^n}{dx^n} (\exp(-x^2)), \quad n = 0, 1, 2, ... \]
Let \( u_n(x) = \exp(-x^2/2)H_n(x) \). Show that \( u_n \) is an eigenfunction of (\(*\)) corresponding to \( \lambda_n = n + 1 \).

Problem 4.
Let \( H \) be a Hilbert space and let \( A : D_A \subset H \to H \) be a linear operator such that \( Au = f \) has a unique solution \( u \in D_A \).
Let \( \{v_i : 1 \leq i \leq n\} \) be linearly independent vectors.
The objective is to find \( u^* \in D_A \) which minimizes \( \|Au - f\|^2 \).
Write \( u^* = \sum_{i=1}^{n} c_i v_i \). Show that \( \{c_i : 1 \leq i \leq n\} \) satisfy
\[ < f, Av_j > = \sum_{i=1}^{n} c_i < Av_i, Av_j >, \quad j = 1, 2, ..., n. \]
Problem 5.

i. Show that the integro-differential equation

\[-u'' + \int_0^1 x y u(y) dy = f, \quad 0 < x < 1\] with boundary conditions \(u(0) = u'(1) = 0\) has a unique solution.

ii. Show that \(\int_0^1 f u dx = \max_{u \in D_A} [2 \int_0^1 f v dx - \int_0^1 (v')^2 dx - \left( \int_0^1 x v dx \right)^2],\)

where \(D_A = \{ v \in C^2[0,1] : v(0) = v'(1) = 0 \} \).

iii. Show that the boundary condition \(v'(1) = 0\) need not be imposed for the variational principle.