Dept of Mathematics and Statistics
King Fahd University of Petroleum & Minerals

AS201: Financial Mathematics

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Final Exam

Monday, May 21, 2012

7.30am-10.00am

Name____ ID#:__________ Serial #:___

Instructions.
1. Please turn off your cell phones and place them under your chair. Any student caught with mobile phones on during the exam will be considered under the **cheating rules** of the University.
2. If you need to leave the room, please do so quietly so not to disturb others taking the test. No two person can leave the room at the same time. No extra time will be provided for the time missed outside the classroom.
3. Only materials provided by the instructor can be present on the table during the exam.
4. Do not spend too much time on any one question. If a question seems too difficult, leave it and go on. Return to it after you attempted other questions.
5. Use the blank portions of each page for your work. Extra blank pages can be provided if necessary. If you use an extra page, indicate clearly what problem you are working on.
6. Only answers supported by work will be considered. Unsupported guesses will not be graded.
7. While every attempt is made to avoid defective questions, sometimes they do occur. In the rare event that you believe a question is defective, the instructor cannot give you any guidance beyond these instructions.
8. Financial calculators, mobile calculators, or communicable devices are disallowed. Use scientific calculator with mathematical equation solving capability or SOA approved financial calculator only. No other materials such as lecture notes, assignments, solution, etc are allowed Write important steps to arrive at the solution of the following problems.

The test is 2 hours and 30 minutes, GOOD LUCK, and you may begin now!

Question	Marks	Comments
1	5	
2	5	
3	10	
4	5	
5	5	
6	8	
7	5	
8 (1-6)	12	
Total	60	

Extra blank pages

1. (5 points) The following table gives the pattern of investment year and portfolio interest rates over a four-year period, where $m = 2$ is the segregation time after which the portfolio method is applicable.

Calendar Year of Original Investment y	Investment i_1^y	Year Rates i_2^y	Portfolio Rates i^{y+2}	Calendar Year of Portfolio Rate $y+2$
2005	5%	6%	8%	2007
2006	4%	5%	6%	2008
2007	2%	4%		
2008	2%			

An investment of 100 is made at the beginning of each of the calendar years 2005, 2006, 2007, and 2008. What is the accumulated value of these deposits at the end of the four-year period?

2. (5 points) Calculate the Macaulay duration (in years) of a 20-year 1000 par value bond which pays 2% coupons semiannually that is redeemed at par. Assume a yield rate of 5% convertible semiannually.

3. (5+5=10 points) Liability payments of 50 each are due to be paid in 2, 3 and 6 years from now. Asset cashflow consists of A_1, in 1 year and A_5 in 5 years. The yield for all payments is 8%. An attempt is made to have the asset cash flow immunize the liability cashflow by matching present value and duration.

a) Find A_1, and A_5.

b) Determine whether or not the conditions for Redington immunization are satisfied.

4. (5 points) Jose and Chris each sell a different stock short for the same price. For each investor, the margin requirement is 50% of the sale price and interest on the margin debt is paid at an effective annual rate of 6%. Each investor buys back his stock one year later at a price of 760. Jose's stock paid a dividend of 32 at the end of the year while Chris's stock paid no dividends. During the 1-year period, Chris's return on the short sale is i, which is twice the return earned by Jose. Calculate i.

5. (5 points) The term structure of effective annual yield rates for zero coupon bonds is given as follows:

1- and 2-year maturity, 10%

3- and 4-year maturity, 12%

You are given the price of a 5-year bond with face amount 100, and annual coupons at rate 5% is 73.68. Find the 4-year forward effective annual interest rate (in effect for the 5th year).

6. (4+4=8 points) Today's term structure of interest rate consists of the following spot rates for maturities of 1 and 2 years

$$s_0(1) = 8\% \qquad\qquad s_0(2) = 10\%$$

Use this term structure to find today's **price** and **yield to maturity** for a 6% annual coupon bond maturing in two years with a face amount of $100.

a) price

b) yield to maturity

7. (5 points) A copier machine costs X and will have a salvage value of Y after 4 years.

(i) Using the straight line method, the annual depreciation expense is 1000.

(ii) Using the declining balance method, the depreciation expense is 33.125% of the book value in the beginning of the year.

Calculate X.

Multiple Choice Questions (2 points each)
Place your answers to the following questions in the grid below.

Question	Answer choice
1	
2	
3	
4	

Question	Answer Choice
5	
6	

1. George invests 5000 in a bank for two years. The investment has a nominal annual interest rate of $x\%$ convertible quarterly during the first year and has a nominal annual discount rate of $x\%$ convertible quarterly during the second year. At the end of the second year, George's account accumulates to 5503.19. Determine x.

A) 2.4 B) 3.2 C) 4.8 D) 6.4 E) 7.2

2. A loan of 4000 is repaid by establishing a sinking fund and making 10 equal payments at the end of each year. The sinking fund earns 6% effective interest rate annually. Immediately after the 4th payment, the yield on the sinking fund unexpectedly increases to 7% effective annually. At that time the payments for the sinking fund are changed to X so that the sinking fund will accumulate to 4000 when it is 10 years after the original loan date. Determine X.

   A) 281 B) 303 C) 321 D) 346 E) 362

3. An n-year 3000 par value bond with 10.8% annual coupons is purchased to yield an annual effective rate j. The book value of the bond at the end of year 11 is 3504.18 and the book value of the bond at the end of year 12 is 3474.53. What is the book value at the end of year 14?

   A) 3378 B) 3393 C) 3401 D) 3408 E) 3415

4. You are given the following information about an investment account:

Date	Value Immediately Before Deposit	Deposit
January 1	100	
Spetember 1	104	X
December 31	210	

Over the year, the dollar-weighted (money-weighted) return is 6%, and the time-weighted return is Y. Calculate Y.

A) 5.92% B) 5.96% C) 6.00% D) 6.04% E) 6.08%

5. A 600 par value 10-year bond with coupons at 6% convertible semiannually can be called on any coupon date starting at the end of year 6. The price of this bond is 500, and the bond is redeemed at par. What is the minimum yield expressed as a nominal annual rate of interest convertible semiannually?

   A) 4.3% B) 4.9% C) 7.7% D) 8.5% E) 9.7%

6. Jason takes out a loan with annual payments at the end of each year. The interest portion of the first payment is 38.30, the principal repaid in the first payment is 47.70, and the outstanding balance just after the first payment is 843.00. What is the annual effective interest rate on Jason's loan?

   A) 4.1% B) 4.2% C) 4.3% D) 4.4% E) 4.5%

END OF TEST PAPER

AS201: Financial Mathematics FORMULA SHEET

Black-Scholes Option pricing model: $d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+(\delta+\frac{\sigma^2}{2})\cdot n}{\sigma\sqrt{n}}$ $\quad d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+(\delta-\frac{\sigma^2}{2})\cdot n}{\sigma\sqrt{n}}$

Price of call option at current time, $C = S_0 \cdot \Phi(d_1) - K \cdot e^{-n\delta} \cdot \Phi(d_2)$

Dividend Discount Model: $P = \sum_{t=1}^{\infty} \frac{d_t}{(1+i)^t} = \sum_{t=1}^{\infty} \frac{d_t}{\prod_{k=1}^{t}(1+i_0(k-1,k))^t} = \sum_{t=1}^{\infty} \frac{d_t}{(1+s_0(t))^t}$

Short selling yield j: $M(1+j) = M(1+i) + S_0 - S_1 - D$

Capital Asset Pricing Model: $\overline{R}_s = \alpha_s + \beta_s \overline{R}_m = R_f + \beta_s(\overline{R}_m - R_f)$

Modified Duration: $DM = -\frac{\frac{d}{di}P}{P} = \frac{\sum_{t=1}^{n} t\cdot K_t\cdot(1+i)^{-t-1}}{P} = \frac{\sum_{t=1}^{n} t\cdot K_t\cdot(1+i)^{-t-1}}{\sum_{t=1}^{n} K_t\cdot(1+i)^{-t}}$

Macaulay Duration: $D = (1+i)\cdot DM = \frac{\sum_{t=1}^{n} t\cdot K_t\cdot(1+i)^{-t}}{P} = \frac{\sum_{t=1}^{n} t\cdot K_t\cdot(1+i)^{-t}}{\sum_{t=1}^{n} K_t\cdot(1+i)^{-t}}$

Modified & Macaulay Duration n-year 0-coupon bond :

$DM = -\frac{\frac{d}{di}P}{P} = \frac{n(1+i)^{-n-1}}{(1+i)^{-n}} = n(1+i)^{-1} = nv$

$D = (1+i)DM = nv(1+i) = n$

Approximate change in present value of a cashflow:

$P(i+h) - P(i) \doteq h\frac{d}{di}P = -h\cdot P(i)\cdot DM$

Macaulay Duration of a n-year coupon bond :

$D = \frac{\sum_{t=1}^{n} t\cdot Fr\cdot v_j^t + n\cdot F\cdot v_j^n}{P} = \frac{\sum_{t=1}^{n} t\cdot Fr\cdot v_j^t + n\cdot F\cdot v_j^n}{\sum_{t=1}^{n} Fr\cdot v_j^t + F\cdot v_j^n} = \frac{Fr\cdot(Ia)_{\overline{n}|} + n\cdot F\cdot v_j^n}{Fr\cdot a_{\overline{n}|} + F\cdot v_j^n}$

Macaulay Duration of a portfolio:

$D = -(1+i)\frac{\frac{d}{di}X}{X} = \frac{\sum_{k=1}^{m} -(1+i)\frac{d}{di}X_k}{X} = \frac{\sum_{k=1}^{m} D_k\cdot X_k}{X}$

Conditions for Redington Immunization:

(i) $PV_A|_{i_0} = PV_L|_{i_0}$ (ii) $\frac{d}{di}PV_A|_{i_0} = \frac{d}{di}PV_L|_{i_0}$ (iii) $\frac{d^2}{di^2}PV_A|_{i_0} > \frac{d^2}{di^2}PV_L|_{i_0}$

Convexity $= \frac{\frac{d^2}{di^2}PV_A|_{i_0}}{PV_A|_{i_0}}$

Portfolio is fully immunized if $\sum A_t v^t \geq \sum L_t v^t$ for any $i > 0$.

PV cashflow $\sum_{k=1}^{n} C_k(1+s_0(t_k))^{-t_k} = C_1(1+s_0(t_1))^{-t_1} + \cdots + C_n(1+s_0(t_n))^{-t_n}$

Price at time 0 t-year 0-coupon bond $= \frac{C}{(1+s_0(t))^t}$

$n-1-$year forward, one year interest rate: $1 + i_0(n-1,n) = \frac{(1+s_0(n))^n}{(1+s_0(n-1))^{n-1}}$

$(1+s_0(n))^n = (1+i_0(0,1))(1+i_0(1,2))\cdots(1+i_0(n-1,n))$

Force of Interest as Forward Rate: $\delta_t = \alpha_t + t\frac{d}{dt}\alpha_t$ $\quad a(t) = e^{t\cdot\alpha_t} = e^{\int_0^t \delta_u du}$

Swap Rate j : $j \times \left[\frac{1}{(1+s_0(1))} + \frac{1}{(1+s_0(2))^2} + \cdots + \frac{1}{(1+s_0(n))^n}\right] = \frac{i_0(0,1)}{(1+s_0(1))} + \frac{i_0(1,2)}{(1+s_0(2))^2} + \cdots + \frac{i_0(n-1,n)}{(1+s_0(n))^n}$

At par yield r_n in $(1+s_0(n))^{-n} + r_n\cdot\sum_{k=1}^{n}(1+s_0(k))^{-k} = 1$

or $\quad r_n = \frac{1-(1+s_0(n))^{-n}}{\sum_{k=1}^{n}(1+s_0(k))^{-k}}$

IRR: Solve for j in $\sum_{k=0}^{n} C_k v_j^{t_k} = 0$

Profitability Index $I = \frac{\text{Present value of cash } \textbf{inflows}}{\text{Present value of cash } \textbf{outflows}}$

Payback Period = first k in $-\sum_{s=0}^{t} C_s \leq \sum_{r=t+1}^{k} C_r$

Discounted Payback Period = first k in $-\sum_{s=0}^{t} C_s v_i^s \leq \sum_{r=t+1}^{k} C_r v_i^r$

Dollar-weighted $I = B - [A + \sum_{k=1}^{n} C_k]$ $\quad i = \frac{I}{A + \sum_{k=1}^{n} C_k(1-t_k)}$

Time-weighted $i = [\frac{F_1}{A} \times \frac{F_2}{F_1+C_1} \times \frac{F_3}{F_2+C_2} \times ... \times \frac{F_k}{F_{k-1}+C_{k-1}} \times \frac{B}{F_k+C_k}] - 1$

Trapezoidal rule to approx $\int_a^b f(x)dx \approxeq \frac{b-a}{2}[f(a)+f(b)]$

Descartes' rule of signs of Polynomial $P(x)$ for counting types of roots:

(i) $n_{+\text{ve roots}} \leq n_{\text{sign changes in } (C_n, C_{n-1}, \ldots, C_1, C_0)}$

(ii) $n_{-\text{ve roots}} \leq n_{\text{sign changes in } ((-1)^n C_n, (-1)^{n-1} C_{n-1}, \ldots, (-1)C_1, C_0)}$

$P = Cv_j^n + Fr \cdot a_{n\rceil j} = C + (Fr - Cj) \cdot a_{n\rceil j} = K + \frac{g}{j}(C - K)$
$P = Fv_j^n + Fr \cdot a_{n\rceil j} = F + F(r - j) \cdot a_{n\rceil j} = K + \frac{r}{j}(C - K)$
(i) $P = F$ Bought at Par (ii) $P > F$ Bought at a Premium (iii) $P < F$ Bought at a Discount
(i) $P = F \leftrightarrow r = j$ (ii) $P > F \leftrightarrow r > j$ (iii) $P < F \leftrightarrow r < j$
$t = \frac{\text{\# of days since last coupon paid}}{\text{\# of days in the coupon period}}$ $P_t = P_0(1+j)^t$ $price_t = P_t - tFr.$
$BV_{t+1} = BV_t(1+j) - Fr$ $I_{t+1} = BV_t \times j$ $PR_{t+1} = Fr - I_{t+1}$
$BV_t = F[1 + (r-j).a_{n-t\rceil}]$ $I_t = F[j + (r-j)(1 - v_j^{n-k+1})]$ $PR_t = F(r-j)v_j^{n-k+1}$

$L = K_1 v + K_2 v^2 + ... + K_n v^n$
$OB_{t+1} = OB_t(1+i) - K_{t+1}$ $I_{t+1} = OB_t \times i$ $PR_{t+1} = K_{t=1} - I_{t+1}$
Retrospective: $OB_t = OB_0(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - ....K_{t-1}(1+i) - K_t$
Prospective: $OB_t = K_{t+1} \times v + K_{t+2} \times v^2 + ... + K_n \times v^{n-t}$
Level payments: $OB_t = L(1+i)^t - Ks_{t\rceil i} = K(a_{n\rceil i}(1+i)^t - s_{t\rceil i}) = K(a_{n-t\rceil})$
$I_t = Kia_{n-t+1\rceil} = K(1 - v^{n-t+1})$ $PR_t = Kv^{n-t+1}$ $PR_t = PR_{t-1}(1+i) = PR_1(1+i)^{t-1}$
Sinking Fund periodic Outlay: $L\left[i + \frac{1}{s_{n\rceil\ j}}\right] = I_t + PR_t$
Sinking Fund periodic Amortization schedule:
$OB_t = L\left[1 - \frac{s_{t\rceil\ j}}{s_{n\rceil\ j}}\right]$ $PR_t = OB_{t-1} - OB_t = L\frac{(1+j)^{t-1}}{s_{n\rceil\ j}}$ $I_t = L \cdot i - L\frac{s_{t-1\rceil\ j}}{s_{n\rceil\ j}} \times j = L\left[i - \frac{(1+j)^{t-1}-1}{s_{n\rceil\ j}}\right]$

Makeham's single loan: $A = Lv_j^n + L_s i a_{n\rceil j} = K + \frac{i}{j}(L - K)$
Makeham's m loans with scheduled repayments:
$A_s = L_s v_j^{t_s} + L_s i a_{t_s\rceil j} = K_s + \frac{i}{j}(L_s - K_s)$ $A = \sum_{s=1}^m A_s = \sum_{s=1}^m \left[K_s + \frac{i}{j}(L_s - K_s)\right] = K + \frac{i}{j}(L - K)$

Accumulated value of n-payment annuity-immediate of 1: $s_{n\rceil i} = \frac{(1+i)^n - 1}{i}$
Present value of n-payment annuity-immediate of 1: $a_{n\rceil}i = \frac{1-v^n}{i}$
Present value of a perpetuity-immediate: $a_{\infty\rceil}i = \frac{1}{i}$
Annuity-due: $\ddot{s}_{n\rceil i} = \frac{(1+i)^n-1}{d}$, $\ddot{a}_{n\rceil i} = \frac{1-v^n}{d}$
Continuous annuities: $\bar{s}_{n\rceil i} = \int_0^n (1+i)^{n-t} dt = \frac{(1+i)^n - 1}{\delta}$, $\bar{a}_{n\rceil i} = \int_0^n v^t dt = \frac{1-v^n}{\delta} = \frac{1-e^{-n\delta}}{\delta}$
Present value of n-term m^{thly} payable annuity-immediate of $1/m$: $a_{n\rceil}^{(m)}i = \frac{1-v_i^n}{i^{(m)}} = a_{n\rceil}i \times \frac{i}{i^{(m)}}$

Present value of annuity with non-level payments: $K_1 v + K_2 v^2 + \cdots + K_{n-1}v^{n-1} + K_n v^n$
Present value of annuity with payments following geometric series:
$v + (1+r)v^2 + \cdots + (1+r)^{n-2}v^{n-1} + (1+r)^{n-1}v^n = \frac{1-(\frac{1+r}{1+i})^n}{i-r}$

a) if $i = r$, then $v + (1+r)v^2 + \cdots + (1+r)^{n-2}v^{n-1} + (1+r)^{n-1}v^n = v + v + \cdots + v = nv$

Accumulated value of annuity with payments following geometric series: $\frac{1-(\frac{1+r}{1+i})^n}{i-r}(1+i)^n = \frac{(1+i)^n - (1+r)^n}{i-r}$
Dividend discount model for present value of a stock: $\frac{K}{i-r}$
n-payment increasing annuity-immediate: $(Is)_{n\rceil} = \frac{\ddot{s}_{n\rceil} - n}{i}$, $(Ia)_{n\rceil} = \frac{\ddot{a}_{n\rceil} - nv^n}{i}$
n-payment increasing perpetuity-immediate: $(Ia)_{\infty\rceil} = \frac{\ddot{a}_{\infty\rceil}}{i} = \frac{1}{id} = \frac{1}{i} + \frac{1}{i^2}$
n-payment decreasing annuity-immediate: $(Ds)_{n\rceil}i = \frac{n(1+i)^n - s_{n\rceil}}{i}$, $(Da)_{n\rceil} = \frac{n - a_{n\rceil}}{i}$
Capitalized Cost: $C = P + \frac{P-S}{i \times S_{n\rceil i}} + \frac{M}{i}$
Periodic Charge = (Capitalized Cost) $\times i = \left(P + \frac{P-S}{i \times S_{n\rceil i}} + \frac{M}{i}\right) \times i = Pi + \frac{P-S}{S_{n\rceil i}} + M$
Depreciation Methods
1) Declining Balance (compound discount) Method: $P_t = P_0 \times (1-d)^t$ $D_t = P_0 \times (1-d)^{t-1} \cdot d$
2) Depreciation - Straight Line Method: $P_t = P_0 - t \times \frac{1}{n} \times (P_0 - P_n)$ $D_t = \frac{1}{n}(P_0 - P_n)$

3) Depreciation - Sum of Year Digits Method: $S_k = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

$$P_t = P_n + \frac{S_{n-t}}{S_n} \times (P_0 - P_n) \qquad D_t = \frac{n-t+1}{S_n} \times (P_0 - P_n)$$

4) Depreciation - Compound interest Method: $P_t = P_0 - \frac{S_{t\rceil i}}{S_{n\rceil i}} \times (P_0 - P_n) \qquad D_t = \frac{(1+i)^{t-1}}{S_{n\rceil i}} \times (P_0 - P_n)$

$$i_{t+1} = \frac{A(t+1)-A(t)}{A(t)}, \qquad A(t) = A(0)a(t)$$

a) $a(t) = (1+i)^t$ Compound interest accumulation factor

b) $a(t) = 1 + it$ Simple interest accumulation factor

$$v = \frac{1}{1+i}, \qquad 1+i = \left[1 + \frac{i^{(m)}}{m}\right]^m \qquad i^{(m)} = m\left[(1+i)^{1/m} - 1\right] \qquad i^{(\infty)} = ln(1+i)$$

$$d = \frac{A(1)-A(0)}{A(1)} = \frac{i}{1+i} \qquad i = \frac{d}{1-d}$$

a) $(1-d)^t$ Compound discount factor

b) $1 - dt$ Simple discount factor

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m \qquad d^{(\infty)} = -ln(1-d)$$

$$\delta_t = \frac{A'(t)}{A(t)} \qquad A(n) = A(0)e^{\int_0^n \delta_t dt}, \qquad i_{real} = \frac{i-r}{1+r}$$

$$1 + x + x^2 + x^3 + \cdots + x^k = \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1}-1}{x-1}$$