1. Under the current market conditions Bond 1 has a price (per 100 of face amount) of 88.35 and a Macaulay duration of 12.7, and Bond 2 has a price (per 100 of face amount) of 130.49 and Macaulay duration of 14.6. A portfolio is created with a combination of face amount $F_1$ of Bond 1 and face amount $F_2$ of Bond 2. The combined face amount of the portfolio is $F_1 + F_2 = 100$, and the Macaulay duration of the portfolio is 13.5. Find $F_1$, $F_2$, and the portfolio value.

\[ P_1 = F_1 \times 88.35/100 \] and \[ P_2 = F_2 \times 130.49/100 \]

The Macaulay duration of the portfolio is \[ \frac{P_1 \times D_1 + P_2 \times D_2}{P_1 + P_2} \]

\[ = \frac{F_1 \times 0.8835 \times 12.7 + F_2 \times 1.3049 \times 14.6}{F_1 \times 0.8835 + F_2 \times 1.3049} = 13.5. \]

So \[ F_1 \times 0.8835 \times 12.7 + F_2 \times 1.3049 \times 14.6 = 13.5(F_1 \times 0.8835 + F_2 \times 1.3049) \]

\[ \rightarrow 11.22F_1 + 19.052F_2 = 11.927F_1 + 17.616F_2 \rightarrow F_2 = 0.49234F_1 \]

Since $F_1 + F_2 = 100$, we solve the two equations to get $F_1 = 67.01$ and $F_2 = 32.99$

The portfolio value is $67.01 \times (0.8835) + 32.99 \times (1.3049) = 102.25$

2. Liability payments of 100 each are due to be paid in 2, 4 and 6 years from now. Asset cashflow consists of $A_1$, in 1 year and $A_5$ in 5 years. The yield for all payments is 10%. An attempt is made to have the asset cash flow immunize the liability cashflow by matching present value and duration.

(a) Find $A_1$, and $A_5$.

(b) Determine whether or not the conditions for Redington immunization are satisfied.

(Work shown 10 points)

(a) \[ \frac{A_1}{1+i} + \frac{A_5}{(1+i)^5} = 100 \left( \frac{1}{(1+i)^2} + \frac{1}{(1+i)^4} + \frac{1}{(1+i)^6} \right) \] where $i_0 = 0.10$.

First condition:

\[ \frac{A_1}{1+i} + \frac{A_5}{(1+i)^5} = 100 \left( \frac{1}{(1+i)^2} + \frac{1}{(1+i)^4} + \frac{1}{(1+i)^6} \right) \rightarrow \frac{A_1}{1+i} + \frac{A_5}{(1+i)^5} = 207.39 \]

\[ \frac{A_1}{1+i} = 207.39 - \frac{A_5}{(1+i)^5} \]

For second condition,

\[ \frac{d}{dt} \left( \frac{A_1}{1+i} + \frac{A_5}{(1+i)^5} \right) = \frac{d}{dt} \left( 100 \left( \frac{1}{(1+i)^2} + \frac{1}{(1+i)^4} + \frac{1}{(1+i)^6} \right) \right) \]

\[ \rightarrow \frac{A_1}{(1+i)^2} - 5 \frac{A_5}{(1+i)^7} = -100 \left( \frac{2}{(1+i)^2} + \frac{4}{(1+i)^4} + \frac{6}{(1+i)^6} \right) \]

Note that with an implied multiplication of 1.1

\[ \rightarrow \frac{A_1}{1+i} + \frac{5A_5}{(1+i)^5} = 100 \left( \frac{2}{(1+i)^2} + \frac{4}{(1+i)^4} + \frac{6}{(1+i)^6} \right) \rightarrow \frac{A_1}{1+i} + \frac{5A_5}{(1+i)^5} = 777.18 \]

\[ \rightarrow 207.39 - \frac{A_5}{(1+i)^5} + \frac{5A_5}{(1+i)^5} = 777.18 \]
Solving for \( A_1 \), and \( A_5 \) results in \( A_1 = 71.44 \) and \( A_5 = 229.41 \),

\[
(b) \sum \frac{d}{dt} (A(1+i)^{-t}) > \sum \frac{d}{dt} (L(1+i)^{-t}) \rightarrow \sum \frac{(t+1)L_i A_i}{(t+1)^{1+t}} > \sum \frac{(t+1)L_i}{(t+1)^{1+t}}
\]

\[
\frac{2A_1}{1.1} + \frac{6.5A_5}{(1.1)^2} = \frac{2}{1.1}(71.44) + \frac{6.5}{(1.1)^2}(229.41) = 3639.1, \quad 100 \left( \frac{3+2}{(1.1)^{1+t}} + \frac{5.4}{(1.1)^{2+t}} + \frac{7.6}{(1.1)^{3+t}} \right) = 3498.1
\]

Since 3639.1 > 3498.1, Redington immunization is satisfied at 10%.

or \( \frac{2A_1}{1.1} + \frac{6.5A_5}{(1.1)^2} = \frac{2}{1.1}(71.44) + \frac{6.5}{(1.1)^2}(229.41) = 4403.3, \quad 100 \left( \frac{3+2}{(1.1)^{1+t}} + \frac{5.4}{(1.1)^{2+t}} + \frac{7.6}{(1.1)^{3+t}} \right) = 4232.7
\]

Since 4403.3 > 4232.7, Redington immunization is satisfied at 10%.

or Simply (as seen in your book), condition 2 and \( \sum \frac{d}{dt} (A(1+i)^{-t}) > \sum \frac{d}{dt} (L(1+i)^{-t}) \rightarrow \sum \frac{A_i^2}{(t+1)^{1+t}} > \sum \frac{L_i t^2}{(t+1)^{1+t}} \)

\[
\frac{A_1}{1.1} + \frac{5^2 A_5}{(1.1)^2} = \frac{1}{1.1}(71.44) + \frac{5^2}{(1.1)^2}(229.41) = 3626.1, \quad 100 \left( \frac{2^2}{(1.1)^{1+t}} + \frac{4^2}{(1.1)^{2+t}} + \frac{6^2}{(1.1)^{3+t}} \right) = 3455.5
\]

Since 3626.1 > 3455.5, Redington immunization is satisfied at 10%.

A note about first and second derivatives versus Redington immunization conditions
(This discussion is NOT in your book or your future AS288 manuals but for those who are intellectually curious)

Condition 1: \( \sum A(1+i)^{-t} = \sum L(1+i)^{-t} \)

Condition 2: \( \sum \frac{d}{dt} (A(1+i)^{-t}) = \sum \frac{d}{dt} (L(1+i)^{-t}) \)

simplifies to \( \sum A_i \frac{t}{(1+i)^{1+t}} = \sum L_i \frac{t}{(1+i)^{1+t}} \)

Condition 3: \( \sum \frac{d}{dt} (A(1+i)^{-t}) > \sum \frac{d}{dt} (L(1+i)^{-t}) \)

simplifies to \( \sum \frac{t^2 A_i}{(1+i)^{1+t}} > \sum \frac{t^2 L_i}{(1+i)^{1+t}} \)

since \( \sum A_i t^2 \) from Redington’s condition no 2.