1. (a) What is the order of

\[ 2xy \frac{d^9y}{dx^9} = 3x^{20} \left( \frac{d^3y}{dx^3} \right)^5 + x + y^{23} - 6 \]

(b) Verify that \( y = -\frac{1}{x+c} \) is a one-parameter family of solutions of the DE \( y' = y^3 \).

(c) Find a solution from the family in (b) that satisfies \( y(0) = -1 \) and determine the largest interval of definition of the solution.

(b) The order of a differential equation is the order of the highest derivative in the equation. Since \( \frac{d^3y}{dx^3} \) is the highest order derivative in the equation, therefore 3 is the order of this differential equation.

(c) \( y = -\frac{1}{x+c} \)

Left-hand side: \( y' = \frac{1}{(x+c)^3} \)

Right-hand side: \( y^3 = \frac{1}{(x+c)^3} \)

Hence \( y' = y^3 \).

(c) \( y = -\frac{1}{x+c} \)

Since \( y(0) = -1 \), therefore we have

\[-1 = -\frac{1}{0+c} \Rightarrow c = 1\]

Thus \( y = -\frac{1}{x+1} \)

It follows from this equation that the largest interval of definition of the solution is \(-1, \infty\).
2. Solve the initial value problem

\[ e^{-2x} \frac{dy}{dx} = (y - y^2)x, \quad y(0) = \frac{1}{2}. \]

By separating the variables, we get

\[ \frac{1}{y - y^2} \, dy = x e^{2x} \, dx. \] (2)

Using partial fraction for \( y \), we get

\[ \left( \frac{1}{y} + \frac{1}{1-y} \right) \, dy = x e^{2x} \, dx. \] (2)

Integrating both sides of this equation, we obtain

\[ \ln(y) - \ln(1-y) = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c. \] (2)

This implies

\[ \ln \left( \frac{y}{1-y} \right) = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c. \] (1)

This gives, by the given initial condition \( y(0) = \frac{1}{2} \);

\[ \ln \left( \frac{1}{2-\frac{1}{2}} \right) = 0 - \frac{1}{4} e + c \implies c = \frac{1}{4}. \] (3)

Thus, the solution of the initial value problem is

\[ \ln \left( \frac{y}{1-y} \right) = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + \frac{1}{4}. \] (3)
3. Solve the ODE

\[ e^x \frac{dy}{dx} - 2xe^y = 2x + 1 \]

It is a linear differential equation in y with standard form

\[ \frac{dy}{dx} - 2xe^x y = (2x + 1)e^{-x} \]

Here \( p(x) = -2x \), \( q(x) = (2x + 1)e^{-x} \), both continuous on \( (-\infty, \infty) \).

So the integrating factor is given by \( \mu(x) = \frac{1}{e^{-2x}} \), \( \frac{d}{dx} \left( e^{-x} y \right) = (2x + 1)e^{-x} \).

Now \( e^{-x} \frac{dy}{dx} - 2xe^{-x} y = (2x + 1)e^{-x} \) implies the equation

\[ \frac{d}{dx} \left[ e^{x^2} y \right] = (2x + 1)e^{-x^2} \]

Integrating both sides of this equation, we have

\[ e^{-x^2} y = \int (2x + 1)e^{-x^2} \, dx \]

By substitution, we get

\[ y = -e^{-(x^2 + x^2)} + C \]

So

\[ y = -e^{-(x^2 + x)} + e^x + Ce^x \]

\[ = -e^x + Ce^{x^2} \text{, } x \in (-\infty, \infty) \]
Question 4: Solve the D.E

\[(2x + y \cos(xy)) \, dx + x \cos(xy) \, dy = 0.\]

Solution: Set \( M = 2x + y \cos(xy) \) and \( N = x \cos(xy). \)

Then:
1. \( \frac{\partial M}{\partial y} = \cos(xy) - xy \sin(xy) \)
2. \( \frac{\partial N}{\partial x} = \cos(xy) - xy \sin(xy) \)
3. \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)

and so the equation is exact.

4. The "implicit" solution is given by \( f(x,y) = C \) where \( f(x,y) = x^2 + \sin(xy) + g(y) \).

(i) \( \frac{\partial f}{\partial x} = M(x,y) \) and (ii) \( \frac{\partial f}{\partial y} = N(x,y) \).

(i): \( \frac{\partial f}{\partial x} = 1 \) \( \Rightarrow \frac{\partial f}{\partial x} = 2x + y \cos(xy) \) \( \Rightarrow f(x,y) = x^2 + \sin(xy) + g(y) \).

(ii): \( \frac{\partial f}{\partial y} = N(x,y) \) \( \Rightarrow x \cos(xy) + g'(y) = x \cos(xy) \).

Then \( g'(y) = 0 \) and \( g(y) = D \).

5. A family of solutions is given by \( x^2 + \sin(xy) = C \).

Similarly, if we start by (ii):

(ii) \( \Rightarrow \frac{\partial f}{\partial y} = N(x,y) \) \( \Rightarrow \frac{\partial f}{\partial y} = x \cos(xy) \).

So \( f(x,y) = \sin(xy) + h(x) \).

Now: (i): \( \frac{\partial f}{\partial x} = M(x,y) \) \( \Rightarrow y \cos(xy) + h'(x) = 2x + y \cos(xy) \).

So \( h'(x) = 2x \) and thus \( h(x) = x^2 \).

5. A family of solutions is given by \( x^2 + \sin(xy) = C \).
Question 5: Find an integrating factor for the D.E. \((xy \cos y - 2x \sin y) \, dy + 2y \sin y \, dx = 0\).

Solution: (i) Determine whether the D.E is exact or Not.

Set \(M(x,y) = \frac{1}{2} y \sin y\) and \(N(x,y) = xy \cos y - 2x \sin y\).

- \(M_y = \frac{\partial M}{\partial y} = 2y \sin y + 2y \cos y\) \(1\) pt
- \(N_x = \frac{\partial N}{\partial x} = y \cos y - 2 \sin y\) \(1\) pt
- \(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\); and so the equation is not exact \(1\) pt.

(ii) Find an integrating factor:

\[
\frac{N_x - M_y}{M} = \frac{y \cos y - 2 \sin y - (2y \sin y + 2y \cos y)}{2y \sin y} = -y \cos y - 4 \sin y
\]

\[
\therefore \text{The integrating factor is } p(y) = e^{-y \cot y - \frac{2}{y}}
\]

\[
\int \frac{N_x - M_y}{M} \, dy = \int \left(-y \cos y - 4 \sin y\right) \, dy = -\frac{1}{2} \ln |\sin y| - 2 \ln |y| + \text{(depend on } y\text{ only!!)} \quad 1\text{ pt}
\]

\(p(y) = e^{-y \cot y - \frac{2}{y}}\) \(1\) pt

(i) \(\int \frac{N_x - M_y}{M} \, dy = \int \left(-\frac{1}{2} \cot y - \frac{2}{y}\right) \, dy = -\frac{1}{2} \ln |\sin y| - 2 \ln |y| + \ln \left(\frac{1}{y \sqrt{|\sin y|}}\right) = \ln \left(\frac{1}{\sqrt{|\sin y|}}\right) - \ln |y| + \ln \left(\frac{1}{y \sqrt{|\sin y|}}\right) = \ln \left(\frac{1}{y^2 \sqrt{|\sin y|}}\right) = \frac{1}{y^2 \sqrt{|\sin y|}} \quad \text{(depend on } x \text{ and } y\text{ !!)}
\]

(ii) \(p(y) = e^{-y \cot y - \frac{2}{y}}\) \(1\) pt

(iii) Notice that: \(\frac{M_y - N_x}{N} = \frac{4 \sin y + y \cos y}{2y^2 \cos y - 2x \sin y}\)

depends on \(x\) and \(y\) !!!
Question 6: Solve the initial value problem:
\[ x \frac{dy}{dx} + (x-2)y = \frac{y^2 \sin x}{x}, \quad y(\pi) = 1. \]

Solution:
1. Put the equation in the standard form by dividing by \( x \).
   \[ \frac{dy}{dx} + \frac{x-2}{x} y = \frac{y^2 \sin x}{x^2}. \]
   \[ \text{1 pt} \]
2. The equation is a Bernoulli's equation with \( n = 2 \).
3. The appropriate substitution is \( u = y^{-1} \), so \( y = u^{-1} \).
   \[ \text{1 pt} \]
4. Then:
   \[ \frac{du}{dx} - \left( \frac{x-2}{x} \right) u = -\frac{\sin x}{x^2} \]
   \[ \text{1 pt} \]
5. Substituting in the equation, we obtain:
   \[ -u^2 \frac{du}{dx} + \frac{x-2}{x} u = -u^2 \frac{\sin x}{x^2} \]
   \[ \text{3 pt} \]
6. Which is a linear equation.

(2) Solve the obtained linear equation:

1. Set \( p(x) = \frac{2x}{x} = \frac{2}{x} - 1 \).
   \[ \text{1 pt} \]
2. The integrating factor is: \( e^\int p(x) dx = e^{-\int \left( \frac{2}{x} - 1 \right) dx} = e^{\ln x} = x \).
   \[ \text{1 pt} \]
3. Multiply the equation by the integrating factor and get:
   \[ \frac{d}{dx} \left[ x^2 e^{-x} u \right] = -x^2 e^{-x} \frac{\sin x}{x^2} \]
   \[ = -e^x \left[ \cos x + \sin x \right] + C \]
   \[ \text{2 pt} \]
4. \[ x^2 e^{-x} u = -\int e^x \sin x dx = \frac{2}{e} \left[ \cos x + \sin x \right] + C \]
   \[ \text{2 pt} \]
5. \[ \frac{1}{y} = u = \frac{e^x}{x^2} \left[ \frac{1}{2} e^x (\cos x + \sin x) + C \right] = \frac{\cos x + \sin x}{2 x^2} + \frac{e^x}{x^2} \]
   \[ \text{2 pt} \]
6. \[ \frac{y}{x^2} = \frac{\cos x + \sin x}{x^2} + De^x \]
   \[ \text{2 pt} \]
7. \[ y = \frac{2x^2}{\cos x + \sin x} + De^x \]
   \[ \text{2 pt} \]
8. \[ y(\pi) = 1 \Rightarrow y(\pi) = 1 \Rightarrow \frac{2\pi^2}{2} + 1 = 1 \]
   \[ \Rightarrow \frac{2\pi^2}{2} = -1 \Rightarrow D = -e^{-2\pi^2 + 1} \]
   \[ \text{2 pt} \]
The solution is 

\[ y = \frac{2x^2}{\cos x + 6x + e^{\pi (2\pi + 1)x^2}} \]

1 pt

**Question 7:** The population of a town grows at a rate proportional to the population present at time \( t \). The initial population of 1000 increases by 30% in 30 years. What will be the population in 40 years?

Solution: \( \frac{dP}{dt} = kP(t) \), where \( P(t) \) is the population at a time \( t \).

- Then \( P(t) = P_0 e^{kt} \)
- So \( P(t) = 1000 e^{kt} \) 2 pts
- \( P = 1000 \) increases by 30% in 30 years
- So \( P(30) = 1000 \cdot \frac{30}{100} + 1000 = 300 + 1000 = 1300 \) 2 pts
- But \( P(30) = 1000 \cdot e^{30k} \)
- Then \( 1300 = 1000 \cdot e^{30k} \)
- So \( e^{30k} = 1.3 \) and then \( 30k = \ln 1.3 \)
- \( \ln 1.3 \)
- Then \( k = \frac{\ln 1.3}{30} \) and \( P(t) = 1000 \cdot e^{\frac{\ln 1.3}{30}t} \) 2 pts
- Finally: \( P(40) = 1000 \cdot e^{\frac{\ln 1.3}{30} \cdot 40} = 1000 \cdot e^{\frac{4}{3} \ln 1.3} \)
- \( = 1000 \cdot (1.3)^{4/3} \) 2 pts
Find the general solution of
\( y(xy+1) \, dx + x(1+xy+x^2y^2) \, dy = 0 \)
using the substitution \( t = xy \)

**Solution: \( t = xy \implies dt = x \, dy + y \, dx \)**
\[\frac{dt}{dx} = \frac{x \, dy + y \, dx}{dx} \implies \frac{dy}{dx} = \frac{1}{x} \left( \frac{dt}{dx} - y \right)\]

On the other hand, from the equation we have
\[\frac{dy}{dx} = -\frac{y(xy+1)}{x(1+xy+x^2y^2)} \times (1+t^2+t^2)\]

Equating (*) and (**), we find
\[\frac{dt}{dx} - y = -\frac{y(t+1)}{1+t+t^2} \implies \frac{dt}{dx} = y \left( 1 - \frac{t+1}{1+t+t^2} \right)\]

\[\frac{dt}{dx} = y \left( \frac{t^2}{1+t+t^2} \right) \implies \frac{dt}{t^2} = \frac{dx}{x} \times \frac{1+t+t^2}{(1+t+t^2)} \times (1+t+t^2)\]

\[\ln|1| = -\frac{1}{2t^2} - \frac{1}{t} + \ln|t| + C\]

Back to \( x \) and \( y \): \( \ln|1x1| = -\frac{1}{2x^2y^2} - \frac{1}{xy} + \ln|xy| + C \)

1pt