Prob. 1 (13 points)
Solve the initial value problem on the interval $(-\infty, 0)$
$4x^2 y'' + y = 0$; $y(-1) = 2$, $y'(-1) = 4$.

Solution: ① Set $t = -x$. Then $x \in (-\infty, 0) \Rightarrow t \in (0, +\infty)$
The D.E. becomes $4t^2 \frac{d^2 y}{dt^2} + y = 0$ with \[ \begin{cases} y(-1) = 2 \\ y'(1) = -4 \end{cases} \]
② Set $y = t^m$. Then $\frac{dy}{dx} = m(m-1)t^{m-2}$.
③ Substituting in the equation (new equation):
we obtain $4m(m-1)t^m + t^m = 0$.
So $(4m(m-1)+1)t^m = 0$.
④ The auxiliary equation is $4m(m-1)+1=0$.
So $4m^2 - 4m + 1 = 0$ and $\Rightarrow (2m-1)^2 = 0$.
Thus $m = \frac{1}{2}$ is a double root. 1 pt
⑤ The solutions are $y_1 = t^{1/2}$ and $y_2 = t^{1/2} \ln t$; and the solution is $y = \int c_1 y_1 + c_2 y_2 = c_1 t^{1/2} + c_2 t^{1/2} \ln t$.

⑥ \[ \begin{cases} y(-1) = 2 \\ \frac{dy}{dt}(-1) = -4 \end{cases} \] \[ \begin{cases} c_1 + c_2 = 2 \\ \frac{1}{2} c_1 + c_2^2 - 4 \end{cases} \] \[ \begin{cases} c_1 = 2 \\ c_2 = -5 \end{cases} \]

⑦ The solution of the initial value problem is
$y = 2t^{1/2} - 5t^{1/2} \ln t = 2(-x)^{1/2} - 5(-x)^{1/2} \ln (-x)$.
**Prob. 2: (13 Points)**
Solve the differential equation

\[ 2y'' - 4y' + 2y = e^x \ln x. \]

**Solution:**

1. Put the equation in the standard form:
   
   \[ y'' - 2y' + y = \frac{1}{2} e^{x \ln x} \]  
   \[ 1 pt \]

2. The homogeneous equation associated to the DE is:
   
   \[ y'' - 2y' + y = 0 \]  
   \[ 1 pt \]

3. The characteristic equation is \( r^2 - 2r + 1 = 0 \).
   
   So \( (r - 1)^2 = 0 \). It has 1 as a double root.  
   \[ 1 pt \]

4. Set \( y_1 = e^x \) and \( y_2 = xe^x \) and use Variation of Parameters.

5. \( W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0 \)  
   \[ 1 pt \]

6. \( W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{1}{2} e^{x \ln x} & e^x + xe^x \end{vmatrix} = -\frac{1}{2} xe^{2x} \ln x \)  
   \[ 1 pt \]

7. \( W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{2} e^{x \ln x} \end{vmatrix} = \frac{1}{2} e^{2x} \ln x \)  
   \[ 1 pt \]

8. \( u_1' = -\frac{1}{2} x \ln x = \frac{w_1}{w} \) \( \Rightarrow \) \( u_1 = \frac{x^2}{8} - \frac{1}{4} x \ln x \)  
   \[ 1 pt \]

9. \( u_2' = \frac{w_2}{w} = \frac{1}{2} \ln x \) \( \Rightarrow \) \( u_2 = \frac{1}{2} x \ln x - \frac{1}{2} x \)  
   \[ 1 pt \]

10. \( y_p = u_1 y_1 + u_2 y_2 = \left( \frac{x^2}{8} - \frac{1}{4} x \ln x \right) e^x + \left( \frac{1}{2} x \ln x - \frac{1}{2} x \right) xe^x \)  
    \[ = \frac{1}{4} x^2 e^{2x} \ln x - \frac{3}{8} x e^{2x} \]  
    \[ 1 pt \]

11. \( y = y_c + y_p = c_1 e^x + c_2 xe^x + \frac{1}{4} x^2 e^{2x} \ln x - \frac{3}{8} x e^{2x} \)  
    \[ 1 pt \]
Prob. 3: (16 Points)
(a) Find three linearly independent functions that are annihilated by the differential operator
\[ D^3 - 8; \quad D = \frac{d}{dx} \]
(b) Use the annihilator approach to solve the differential equation
\[ y'' - 9y = 2e^{5x} - 8\cos(2x) \]
(Do not evaluate the constants!)

Solution:
\[ D^3 - 8 = (D - 2)(D^2 + 2D + 4) \]
So the charact. equation would be \( r^2 - 2r + 2r + 4 = 0 \)
So \( r = 2 \) and \( r = -1 \pm i\sqrt{3} \)
Therefore the required linearly independent functions are,
\[ e^{2x}, \quad e^{-x}\cos(\sqrt{3}x), \quad e^{-x}\sin(\sqrt{3}x) \]

Annihilator approach to solve \( y'' - 9y = 2e^{5x} - 8\cos(2x) \)
Annihilator for \( e^{5x} \) is \( (D - 5) \)
Annihilator for \( \cos(2x) \) is \( (D^2 + 4) \)
Then we obtain for \( 2e^{5x} - 8\cos(2x) \) is \( (D - 5)(D^2 + 4) \)
The corresponding char. equation is \( (r^2 - 9)(r - 5)(r^2 + 4) \)
The roots are \( r = 3, -3, 5, 2i, -2i \)
Thus the general solution is
\[ y = c_1 e^{3x} + c_2 e^{-3x} + c_3 e^{5x} + c_4 \cos(2x) + c_5 \sin(2x) \]
**Prob. 4: (14 Points)**

Solve the initial value problem

\[ y'' - 5y' + 100y - 500y = 0; \ y(0) = 0, \ y'(0) = 10, \ y''(0) = 250 \]

given that \( y_1(x) = e^{5x} \) is a solution of the differential equation.

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**Solution:**

1. The characteristic equation is \( r^3 - 5r^2 + 100r - 500 = 0 \)

2. Since \( e^{5x} \) is a solution for the D.E., \( r = 5 \) must be a root of the "characteristic equation".

3. By long division \( r^3 - 5r^2 + 100r - 500 = \frac{(r - 5)(r^2 + 100)}{1} \)

   and so the roots are \( r = 5, \ r = \pm 10i \)

4. The general solution is given by
   \[
   y = C_1 e^{5x} + C_2 \cos(10x) + C_3 \sin(10x)
   \]

5. \[
   y' = 5C_1 e^{5x} - 10C_2 \sin(10x) + 10C_3 \cos(10x)
   \\
   y'' = 25C_1 e^{5x} - 100C_2 \cos(10x) - 100C_3 \sin(10x)
   \\
   \begin{align*}
   y(0) &= 0 \\
   y'(0) &= 10 \\
   y''(0) &= 250
   \end{align*}
   \]

   \[
   \begin{cases}
   C_1 + C_2 = 0 \\
   5C_1 + 10C_3 = 10 \\
   25C_1 - 100C_2 = 250
   \end{cases} \implies \begin{cases}
   C_1 + 2C_3 = 2 \\
   C_1 - 4C_2 = 10
   \end{cases} \implies \begin{cases}
   C_1 = -2 \\
   C_2 = -2
   \end{cases} \implies C_3 = 0
   \]

6. The solution is
   \[
   y = 2e^{5x} - 2 \cos(10x)
   \]
Prob. 5: (11 Points)

Let \( y_1 = x^{-1/2} \sin x \) be a solution of \( x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \). Use the reduction of order method to find a second solution.

Write given D.E. in standard form
\[
y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right)y = 0, \quad x \neq 0
\]

The second solution is given by the reduction of order method.

\[
y_2 = y_1(x) \int \frac{e^{-\frac{1}{2} \log x} \, dx}{(y_1(x))^2}
\]

\[= x^{-\frac{1}{2}} \sin x \int \frac{-\frac{1}{2} \log x \, dx}{(x^{-\frac{1}{2}} \sin x)^2}
\]

\[= x^{-\frac{1}{2}} \sin x \int \frac{-\frac{1}{2} \log x}{x^{-1} \sin^2 x} \, dx
\]

\[= x^{-\frac{1}{2}} \sin x \int \cos^2 x \, dx
\]

\[= -x^{-\frac{1}{2}} \sin x \cos x = -x^{-\frac{1}{2}} \sin x \frac{\cos x}{\sin x}
\]

\[= -x^{-\frac{1}{2}} \cos x
\]
**Prob. 6: (11 Points)**

Consider the differential equation

\[ y'' + y = \sec x + e^x \]

(a) Check that \( x \sin x + (\cos x) \ln(\cos x) \) is a particular solution of

\[ y'' + y = \sec x. \]

(b) Find the general solution of \( y'' + y = \sec x + e^x \).

\[ \text{Let } y_p = x \sin x + \cos x \ln(\cos x) \]

Then

\[ y'_p = \sin x + x \cos x + \cos x \frac{\cos x}{\cos x} - \frac{\sin x}{\cos x} \]

\[ y''_p = \cos x + \cos x \ln(\cos x) + x \sin x + \cos x \ln(\cos x) \]

So

\[ y'' + y = \cos x + \cos x \ln(\cos x) + \frac{x \sin x + \cos x \ln(\cos x)}{\cos x} = \sec x \]

If we look for a particular solution of \( y'' + y = e^x \) in the form \( y_p = Ce^x \), then we have

\[ Ce^x + Ce^x = e^x \Rightarrow 2Ce^x = e^x \Rightarrow C = \frac{1}{2}. \]

So

\[ y_p = \frac{1}{2} e^x. \]

The general solution of the homogeneous equation \( y'' + y = 0 \) is

\[ y_g = C_1 \cos x + C_2 \sin x. \]

Hence, by superposition principle, we have

\[ y = C_1 \cos x + C_2 \sin x + x \sin x + x \sin x + \cos x \ln(\cos x) + \frac{1}{2} e^x. \]
**Prob. 7: (11 Points)**

Show that \( x, x \ln x \) and \( x^2 \) form a fundamental set of solutions (are solutions and are linearly independent) of the differential equation

\[
x^3 y''' - x^2 y'' + 2xy' - 2y = 0, \quad x > 0.
\]

*First we verify that \( y_1 = x \), \( y_2 = x \ln x \) and \( y_3 = x^2 \) are solutions of the differential equation (D.E.).*

\[
y_1 = x : D.E. \Rightarrow x^3 (0) - x^2 (0) + 2x (1) - 2x = 0
\]

\[
y_2 = x \ln x : \quad y_2' = \ln x + 1, \quad y_2'' = \frac{1}{x}, \quad y_2''' = \frac{1}{x^2}
\]

\[
D.E. \Rightarrow x^3 \left( -\frac{1}{x^2} \right) - x^2 \left( \frac{1}{x} \right) + 2x (\ln x + 1) - 2x \ln x = 0
\]

\[
y_3 = x^2 \quad y_3' = 2x, \quad y_3'' = 2, \quad y_3''' = 0
\]

\[
D.E. \Rightarrow x^3 (0) - x^2 (2) + 2x (2x) - 2x^2 = 0
\]

*To check linear independence of the functions, we calculate their Wronskian \( W \):*

\[
W = \begin{vmatrix}
x & x \ln x & x^2 \\
1 & \ln x + 1 & 2x \\
0 & \frac{1}{x} & 2
\end{vmatrix}
\]

\[
= x \left[ 2x \ln x + 2 - 2 - \frac{1}{x} \right] - 1 \left[ 2x \ln x - x \right] + 0
\]

\[
= 2x \ln x + x - 2x \frac{\ln x}{x} = x \neq 0 \text{ in } (0, \infty).
\]

Thus \( x, x \ln x \) and \( x^2 \) form a fundamental set of solutions.
Prob. 8: (11 Points)

Let \( y = C_1 \cos \omega x + C_2 \sin \omega x, \omega \neq 1 \), be a 2-parameter family of solutions of the differential equation \( y'' + \omega^2 y = 0 \). Determine whether a member of the family can be found that satisfies the boundary conditions \( y(0) = 1 \) and \( y'(\frac{\pi}{2\omega}) = -1 \).

Suppose a member of the family can be found that satisfies the boundary conditions.

\[
y = c_1 \cos \omega x + c_2 \sin \omega x, \quad \omega \neq 1
\]

\[
y' = -c_1 \omega \sin \omega x + c_2 \omega \cos \omega x
\]

\[
y(0) = 1 \Rightarrow 1 = c_1 + c_2(0) \Rightarrow c_1 = 1
\]

\[
y'(\frac{\pi}{2\omega}) = -1 \Rightarrow -1 = -c_1 \omega \sin (\frac{\pi}{2\omega}) + c_2 \omega \cos (\frac{\pi}{2\omega})
\]

\[
-1 = -c_1 \omega
\]

\[
c_1 = \frac{1}{\omega}
\]

where \( \omega \neq 1 \)

This contradicts \( c_1 = 1 \).

So the given boundary-value problem has no solution.