Exercise 1 (10 points).
Let $V = \mathbb{R}^3$ and $S = \{U_1, U_2, U_3\}$ its standard basis.
(1) Find a basis $T = \{V_1, V_2, V_3\}$. $T \neq S$ and a $3 \times 3$ matrix $A$ such that
\{AV_1, AV_2, AV_3\} is not a basis for $V$.
(2) Find a $3 \times 3$ matrix $A$ such that \{AV_1, AV_2, AV_3\} is a basis for $V$. 
Exercise 2 (10 points).
Let $V$ be the vector space of all continuous real-valued functions and $W$ the subspace of $V$ spanned by \{cost, sint, t\}.
(1) Find a basis for $W$.
(2) Find $\dim W$. 
Exercise 3 (10 points).
Let $V = \mathcal{M}_{n \times n}(\mathbb{R})$ be the vector space of all $n \times n$ matrices and $A$ a fixed $n \times n$ matrix. Let $\phi_A : V \to \mathbb{R}$ defined by $\phi_A(B) = Tr(AB)$ where $Tr(M)$ is the trace of the matrix $M$.
(1) Prove that $\phi_A$ is a homomorphism.
(2) Is $\phi_A$ an isomorphism?
Exercise 4 (10 points).
Let $V$ and $W$ be two vector spaces over a field $K$ and let $\phi : V \to W$ be an isomorphism.

(1) Prove that $\phi^{-1} : W \to V$ is an isomorphism.
(2) Prove that if $S = \{U_1, \ldots, U_n\}$ is a basis for $V$, then $\phi(S) = \{\phi(U_1), \ldots, \phi(U_n)\}$ is a basis for $W$.
(3) Prove that $\dim V = \dim W$. 
Exercise 5 (10 points).
Let \( V = \mathcal{M}_{n \times n}(\mathbb{R}) \) be the vector space of all \( n \times n \) matrices and \( W \) the subset of \( V \) consisting of all symmetric matrices.
(1) Prove that \( W \) is a vector space.
(2) Find \( \dim W \).
(3) As an application, Find a basis for \( W \) when \( n = 3 \).
Exercise 6 (10 points).

Let $V = \mathbb{P}_2 = \{ f \in K[X] | \deg(f) \leq 2 \}$ and set $S = \{1, X, X^2\}$ and $T = \{1 - X, 1 + X, 1 - X^2\}$.

(1) Prove that $S$ and $T$ are bases for $V$.
(2) Find the transition matrix from $S$ to $T$. 
Exercise 7 (10 points).
Let $V = \mathbb{R}^3$, $S = \{U_1 = (0, 1, 1), U_2 = (1, 0, 1), U_3 = (1, 1, 0)\}$ be a basis for $V$ and set $V_1 = (1, -1, 1), V_2 = (0, 1, 2)$ and $V_3 = (0, 0, 2)$.

1. Verify that $\{V_1, V_2, V_3\}$ are linearly independent.
2. Verify that the coordinate vectors $[V_1]_S, [V_2]_S$, and $[V_3]_S$ are linearly independent.

Generalizations. Prove that if $W$ is an $n$-dimensional vector space, $T$ is a basis for $W$ and $W_1, \ldots, W_n$ are linearly independent, then the coordinate vectors $[W_1]_T, \ldots, [W_n]_T$ are linearly independent.
Exercise 8 (10 points).
Let $A$ be an $n \times n$ non-singular matrix and $B$ be an $n \times m$ matrix with $\text{rank}(B) = p$.

(1) Find $\text{rank}(AB)$.

(2) Set $M = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 5 \\ 3 & 2 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 3 & 0 \end{pmatrix}$

Find $\text{rank}(N)$ and $\text{rank}(MN)$. 
Exercise 9 (10 points).
Let $V$ be the vector space of all real-valued continuous functions, $V_{even}$ its subspace of all even functions and $V_{odd}$ its subspace of all odd functions.

1) Prove that every element $f \in V$ can be written as $f = f_1 + f_2$ where $f_1 \in V_{even}$ and $f_2 \in V_{odd}$.

2) Prove that $V_{even} \cap V_{odd} = \{0\}$. 
Exercise 10 (10 points).
Let $V$ be the vector space of all real-valued continuous functions on the closed interval $[a, b]$ and consider the map $\phi : (V \times V) \rightarrow \mathbb{R}$ defined by $\phi(f, g) = \int_a^b f(t)g(t)dt$.
(1) Prove that $\phi$ is an inner product.
(2) Find the distance from $f(t) = t$ to $g(t) = e^t$. 