Exercise 1 (20 points)
Use the augmented matrix to find all values of $r$ for which the system (S) has:
\( \begin{cases} 
2x + y + z = 2 \\
-x + y + r^2z = r \\
x + y - z = 2 
\end{cases} \)

\text{a/ No solution} \quad \text{b/ a unique solution} \quad \text{c/ infinitely many solutions}
Exercise 2 (20 points)

Let \( L : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \), \( L \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) = \left( \begin{array}{c} c \\ b-d \\ a \\ d-b \end{array} \right) \)

2- Find \( \text{Ker}L \) and \( \dim \text{Ker}L \). Is \( L \) a one-to-one linear operator?

3- Find \( \text{Range}L \) and \( \dim(\text{Range}L) \). Is \( L \) an onto linear operator?
Exercise 3 (20 points)
Let $P_n$ be the vector space of all real polynomials of degree $\leq n$ and $D : P_4 \to P_3, D(f) = f'$ be the differential operator. Let $S = \{1, t, t^2, t^3, t^4\}$ and $T = \{1, t, t^2, t^3\}$ be the standard bases of $P_4$ and $P_3$ respectively. Set $S' = \{2, 1-t, t^2-t, t^3-t^2, t^4-t^3\}$ and $T' = \{1, 1-t, 1-t^2, t^3\}$.

1- Prove that $S'$ and $T'$ are bases of $P_4$ and $P_3$ respectively.
2- Find the transition matrix $P$ from $S'$ to $S$.
3- Find the transition matrix $Q$ from $T'$ to $T$.
4- Find the matrix representing $D$ with respect to $S'$ and $T'$.
Exercise 4 (20 points)

Use Gram-Schmidt process to transform the basis $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ to an orthonormal basis.
Exercise 5 (20 points)

Let \( A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) and set \( W = \{ B \in M_3(\mathbb{R}) \mid AB = BA \} \).

1- Prove that \( W \) is a vector space.
2- Find a basis for \( W \) and \( \dim W \).
3- Find the value of the positive integer \( n \) such that \( W \) is isomorphic to \( \mathbb{R}^n \).
Exercise 6 (20 points)

Let $A$ and $B$ be two $n \times n$ similar matrices (i.e. $B = P^{-1}AP$).
1-Prove that if $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $A$ and $V_1$ and $V_2$ are eigenvectors associated to $\lambda_1$ and $\lambda_2$ respectively, then $V_1$ and $V_2$ are linearly independent.
2-Prove that $A$ and $B$ have the same eigenvalues.
3-Prove that for a common eigenvalue $\lambda$ of $A$ and $B$, if $V$ is an eigenvector of $A$ associated to $\lambda$, then $P^{-1}V$ is an eigenvector of $B$ associated to $\lambda$. 
Exercise 7 (20 points)

Let $A$ be an $n \times n$ matrix with characteristic polynomial $f = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ and suppose that $a_0 \neq 0$.

1-Prove that $A$ is invertible and find $A^{-1}$.

2-Application: Find the $3 \times 3$ matrix $A$ such that $f = X^3 - 4X^2 + 4X - 1$ and $A^2 - 4A = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -3 & 0 \\ -1 & 0 & -3 \end{pmatrix}$
Exercise 8 (20 points)
Let $L$ be a linear operator of $\mathbb{R}^3$ whose matrix in the standard basis $S$ is

$$[L]_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

1-Prove that $L$ is diagonalizable.
2-Find an orthonormal basis $T$ such that $[L]_T = D$ is a diagonal matrix.
Exercise 9 (20 points)
Let $A$ be a real symmetric matrix.
1-Prove that the eigenvalues of $A$ are all real numbers.
2-Prove that eigenvectors associated to distinct eigenvalues are orthogonal.
Exercise 10 (20 points)
Let \( g(x, y, z) = x^2 + y^2 + z^2 + 4xy + 4xz - 4yz \) be a quadratic form of \( R_3 \).
1-Find the canonical quadratic form \( h \) that is equivalent to \( g \).
2-Find the rank and the signature of \( g \).