Department of Mathematics and Statistics

Final Exam for Math 470

Semester 2, Academic year 2011-2012

Time allowed: Two hours

Full Name: ....

ID Number: ....

Question 1 Use the method of separation of variables to solve the BVP:

$$\begin{cases} u_t = u_{xx}, \quad -\pi < x < \pi, \quad t > 0 \\ u(\pi, t) - u(-\pi, t) = 0, \quad t > 0 \\ u_x(\pi, t) - u_x(-\pi, t) = 0, \quad t > 0 \\ u(x, 0) = f(x), \quad -\pi < x < \pi. \end{cases}$$

Question 2 Solve

$$\begin{cases} \Delta u(x, y) = 0 & \text{for } \; x > 0, \; y > 0 \\ u(0, y) = 0 & \text{for } \; y > 0 \\ u(x, 0) = x & \text{for } \; x > 0\,. \end{cases}$$

Question 3 Find a bounded solution of the following problem:

$$\begin{cases} u_t = 4u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) = \mathrm{e}^{-|x|}, \quad -\infty < x < \infty. \end{cases}$$

Question 4 Solve

$$\begin{cases} \Delta u(x, y) = 0 & \text{for } \; 0 \le x^2 + y^2 < 9 \\ \partial_n u(x, y) = 4xy & \text{for } \; x^2 + y^2 = 9\,. \end{cases}$$

Question 5 Use Laplace transform to solve

$$\begin{cases} u_{xx} - 2u_x - u_{tt} = 0, \quad x > 0, \quad t > 0 \\ u(0, t) = 0, \quad t > 0 \\ u(x, 0) = 1, \quad u_t(x, 0) = 0, \quad x > 0. \end{cases}$$

Question 6 Consider the following problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain with piecewise smooth boundaries and f is a nice function.

a) Show that the solution u of (1) satisfies

$$\int_\Omega [u\,v + \nabla u \nabla v] d\mathbf{x} = \int_\Omega f\,v d\mathbf{x} \quad \text{for all} \quad v \in H^1(\Omega) \tag{2}$$

where the Hilbert space

$$H^1(\Omega) = \{v \in L_2(\Omega); \int_\Omega v^2 d\mathbf{x} < \infty \text{ and } \int_\Omega (\nabla v)^2 d\mathbf{x} < \infty\}.$$

b) Show that (1) has a unique weak solution u in $H^1(\Omega)$.