King Fahd University of Petroleum and Minerals  
Department of Mathematics and Statistics  

MATH 513 Mathematical Methods for Engineers  
Exam 2  
Term 112  
Section 01 Instructor: Dr Nadeem Malik  

Thursday 24\textsuperscript{th} May 2012  
Room 5-102  
Time allowed: 7pm – 10pm (3 hours)  

Important Instructions  

1. Write your name and ID number on each sheet that you use.  
2. At the end of the exam, place all your answer sheets in good order in a bundle. The instructor will staple these together.  
3. Calculators are allowed, but not programmable calculators.  
4. Mobiles must be switched off at all times during the exams; they must be placed in front of the student at all times.  
5. Food is not allowed. Drinks are allowed.  
6. Answer all questions. Part I (short questions) is worth a total of 30 points; Part II (long questions) is worth a total of 80 points.  

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513 Exam 2, Questions

Part I (roughly 1 hour)
(10 points each question)

1. (a) Find the characteristic polynomial, the eigenvalues and the associated eigenvectors of the matrix,

\[ A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}. \]

(b) Hence using linear algebra methods only, solve the differential equation,

\[ \frac{dX}{dt} = AX \]

where \( X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \), and the initial condition is, \( X(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \).

2. Use Cramer’s rule to solve the system,

\[
\begin{align*}
3x_1 + 2x_2 + x_3 &= 7 \\
x_1 - x_2 + 3x_3 &= 3 \\
5x_1 + 4x_2 - 2x_3 &= 1
\end{align*}
\]

3. Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem,

\[ y'' + \mu y = 0, \quad 0 < x < \pi \]

subject to \( y'(0) = 0, \ y(\pi) - y'(\pi) = 0 \).
Part II (roughly 2 hours)
(18—22 points each question)

4. Use the method of separation of variables to solve,

\[ u_t = 9u_{xx}, \quad 0 < x < 1, \quad t > 0 \]

with boundary conditions:
\[ \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0, \quad t > 0; \]
and initial condition:
\[ u(x, 0) = 1 + 3\cos(\pi x) - 2\cos(3\pi x), \quad 0 < x < 1. \]

[18 points]

5. Consider the wave equation,

\[ \frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0 \]

with boundary condition \( u(0, t) = 0, \ u(2, t) = 3 \), and
initial conditions \( u(x, 0) = 0 \) and \( \frac{\partial u}{\partial t}(x, 0) = 0 \).

(a) Show that the Laplace transform of \( u \) is,
\[ U(x, s) = 3 \sinh(sx/2)/(s \sinh(s)) \]

(b) Re-express \( U(x, s) \) in terms of negative exponentials.

(c) Expand the denominator in \( U(x, s) \) using the binomial theorem to
obtain a new expansion for \( U(x, s) \) in terms of sums of negative
exponentials.

(d) Hence show that the solution is,
Note,

**Binomial theorem:** \((1 + a)^p = 1 + pa + \frac{p(p-1)}{2!} a^2 + \frac{p(p-1)(p-2)}{3!} a^3 + \cdots, \) for \(|a| < 1,\) and for any real number \(p.\)

[20 points]

6. Use the separation of variables method to solve the Laplace equation for \(u(r, \theta)\) in spherical coordinates,

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial u}{\partial \theta} \right) = 0, \quad 0 < r < 1, \quad 0 < \theta < \pi
\]

if

\[
\begin{align*}
    u(1, \theta) &= 2, & 0 < \theta < \pi/2 \\
    u(1, \theta) &= -2, & \pi/2 < \theta < \pi
\end{align*}
\]

Furthermore, the solution remains finite at \(r = 0.\)

(Find the first 3 terms only of the expansion for \(u(r, \theta).\))

*Hints: (i) Change the independent variable using, \(r = e^{-s}.\) (ii) Use the substitution \(\mu = \cos \theta.\)"

[22 points]
7. Solve the Poisson’s equation,

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad 0 < x < 1, \quad 0 < y < 2
\]

if \( u(0, y) = u(1, y) = u(x, 0) = u(x, 2) = 0 \), and

\[
f(x, y) = \sin(\pi x) \sin\left(\frac{\pi y}{2}\right) - 2 \sin(2\pi x) \sin\left(\frac{3\pi y}{2}\right).
\]

[Hint, first solve the Poisson’s equation with \( f(x, y) = \lambda u \).]

[20 points]