King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 536 [Functional Analysis II]
Second Semester 2011-2012 (112)

Exam I: March 14, 2012, **Time 2 hours:**

Q1. (a) Let H be an infinite dimensional Hilbert space. Use an appropriate result to prove that H has an infinite orthonormal subset.

(b) Consider the complete orthonormal sequence $\varphi_1(x) = \frac{1}{\sqrt{2\pi}}, \varphi_{2n}(x) = \frac{1}{\sqrt{\pi}} \sin nx, \varphi_{2x+1} = \frac{1}{\sqrt{\pi}} \cos nx$ in $L_2[-\pi, \pi]$. Apply parseval's formula to the function $f(x) = x \ (-\pi \leq x \leq \pi)$ and show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Q2. (a) Let H be a Hilbert space and $T \in L(H)$, the space of all bounded linear Operators on H. Prove that there exists a unique operator $T^* \in L(H)$ such that $\langle Tx, y \rangle = \langle x, T_y^* \rangle$ for all $x, y \in H$.

(b) Suppose that H is a complex Hilbert space and $T \in L(H)$. Then show that T is self-adjoint if and only if $\langle Tx, x \rangle$ is real for each $x \in H$.

Q3. (a) Let T be a bounded linear operator on a Hilbert space H. If T is normal, then show that $\|T^2\| = \|T\|^2$

(b) If S and T are positive operators on a Hilbert space such that $ST = TS$, then prove that ST is positive.

Q4. (a) Let X be a normed space with dual X^*. Prove that the closed ball $B_1^* = \{f \in X^* : \|f\| \leq 1\}$ is compact with respect to weak*-topology.

(b) Prove that every separable infinite-dimensional Hilbert space is linearly isometric to $(l_2, \|\cdot\|_2)$.