Q1. (a) Let $P$ be an orthogonal projection on an inner product space $X$. If $N(P)$ is the null space of $P$ and $R(P)$ is the range of $P$, then show that $N(P) = [R(p)]^\perp$.

(b) If $P$ is a bounded linear projection on a Hilbert space $H$, then show that $P$ is self-adjoint and idempotent.

Q2. (a) Let $P_1$ and $P_2$ be projections on a Hilbert space $H$. If $P = P_1 + P_2$ is a projection, then prove that $P$ projects $H$ onto $Y = Y_1 \oplus Y_2$ where $P_1(H) = Y_1$ and $P_2(H) = Y_2$.

(b) Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. For $x \in H, z \in K$ is a projection of $x$ if and only if $\langle x - z, y - z \rangle \leq 0$ for all $y \in K$. Use this fact to prove that the projection operator $P_K(x)$ of $H$ onto $K$ satisfies $\| P_K(u) - P_K(v) \| \leq \| u - v \|$ for all $u, v \in H$.

Q3. (a) Let $\{T_n\}$ be a sequence of compact linear operators from a normed space $X$ into a Banach space $Y$. If $\{T_n\}$ is uniformly operator convergent to $T$ (i.e. $\| T_n - T \| \to 0$ as $n \to \infty$), then prove that limit operator $T$ is compact.

(b) Let $T: l_2 \to l_2$ be defined by

$$T(x) = \frac{\xi_j}{j} \text{ where } x = \{\xi_j\} \in l_2.$$  

Use above part (a) to show that $T$ is a compact operator.

Q4. (a) Let $X$ and $Y$ be normed spaces and $T \in BL(X,Y)$, the space of all bounded linear operators from $X$ into $Y$. Define Banach adjoint $T^X$ of $T$ from $Y^*$ to $X^*$. Use an appropriate consequences of the Hahn–Banach theorem to prove that $\| T^X \| = \| T \|$. 

(b) Let $X$ and $Y$, be normed spaces. If $T$ is a compact linear operator from $X$ into $Y$, then verify that $T^X$ is also compact.