1. (a) Let $H$ be a Hilbert space and $T \in L(H)$, the space of bounded linear operators from $H$ into $H$. Define adjoint $T^*$ of $T$. Hence verify that the left shift operator is adjoint of the right adjoint shift operator on the usual Hilbert space ($l^2$, $< \cdot , \cdot >$).

(b) If $T$ is as in part (a), then show that $\| T \| = \sup \{ |< Tx, x >| : \| x \| \leq 1 \}$ provided $T$ is a self-adjoint.

2. (a) Define a unitary operator on a Hilbert space $H$. Give an example to show that a normal operator on $H$ may not be unitary.

(b) Let $T$ be a one-one linear operator on a Hilbert space $H$ such that $T^{-1}$ exists and $D(T) = D(T^{-1}) = H$. Then show that $T^*$ is one-to-one and $(T^*)^{-1} = (T^{-1})^*$.

3. (a) Show by means of example, that in the usual normed space ($l^1$, $\| \cdot \|$), a weak* convergent sequence may not be weak convergent. Also justify the statement that every weakly Cauchy sequence in $l^1$ is norm convergent.

(b) Prove that $P = P_1P_2$ is a projection on a Hilbert space $H$ if the projection $P_1$ and $P_2$ commute on $H$.

4. (a) Show that two closed subspaces $Y$ and $V$ of a Hilbert space $H$ are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$

(b) Let $X$ and $Y$ be normed spaces and $T : X \rightarrow Y$ a linear operator. If $\dim X < \infty$, then prove that $T$ is compact.

5. (a) Let $X$ be a complex Banach space and $T$ a bounded linear operator from $X$ into $X$. Then prove that the spectrum of $T$, $\sigma(T)$, is a compact subset of $\mathbb{C}$ and $\sigma(T) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq \| T \| \}$.

(b) Define $T : l^2 \rightarrow l^2$ by $T(x) = \{0, x_1, x_2, \ldots \}$ where $x = \{x_1, x_2, \ldots \} \in l^2$. Use part (a) to show that $\sigma(T) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$.

6. (a) Prove that the spectrum $\sigma(T)$ of a bounded linear and self-adjoint operator $T : H \rightarrow H$ (complex Hilbert space) is real.

(b) Let $T$ be a compact self-adjoint linear operator on a complex Hilbert space $H$. Prove that there is an eigen value $\lambda$ of $T$ such that $|\lambda| = \| T \|$ and there is a corresponding eigen vector $x$ such that $Tx = \lambda x$, $\| x \| = 1$, and $|< Tx, x >| = \| T \|$.