Kind Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 536 [Functional Analysis II]
Second Semester 2011-2012 (112)

Final Exam: May 19, 2012 Time: 3 hours

1. (a) Let H be a Hilbert space and $T \in L(H)$, the space of bounded linear operators from H into H. Define adjoint T^* of T . Hence verify that the left shift operator is adjoint of the right adjoint shift operator on the usual Hilbert space ($l_2, < \cdot >$).
(b)If T is as in part (a), then show that
$\| T \| = \sup\{ | < Tx, x > | : \| x \| \leq 1\}$ provided T is a self-adjoint

2. (a) Define a unitary operator on a Hilbert space H. Give an example to show that a normal operator on H may not be unitary.
(b) Let T be a one-one linear operator on a Hilbert space H such that T^{-1} exists and $\overline{D(T)} = \overline{D(T^{-1})} = H$. Then show that T^* is one-to-one and $(T^*)^{-1} = (T^{-1})^*$.

3. (a) Show by means of example, that in the usual normed space $(l_1, \| \cdot \|)$, a weak* convergent sequence my not be weak convergent. Also justify the statement that every weakly Cauchy sequence in l_1 is norm convergent.
(b) Prove that $P = P_1 P_2$ is a projection on a Hilbert space H if the projection P_1 and P_2 commute on H.

4. (a) Show that two closed subspaces Y and V of a Hilbert space H are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$
(b) Let X and Y be normed spaces and $T : X \to Y$ a linear operator.
If dim $X < \infty$, then prove that T is compact.

5. (a) Let X be a complex Banach space and T a bounded linear operator from X into X. Then prove that the spectrum of $T, \sigma(T)$, is a compact subset of $\mathbb{C}$ and $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \| T \|\}$.
(b) Define $T : l_2 \to l_2$ by $T(x) = \{0, x_1, x_2, \ldots\}$ where $x = \{x_1, x_2, \ldots\} \in l_2$. Use part (a) to show that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

6. (a) Prove that the spectrum $\sigma(T)$ of a bounded linear and self-adjoint operator $T : H \to H$ (complex Hilbert space) is real.
b) Let T ba a compact self-adjoint linear operator on a complex Hilbert space H. Prove that there is an eigen value λ of T such that $|\lambda| = \| T \|$ and there is a corresponding eigen vector x such that
$Tx = \lambda x, \| x \| = 1$, and $| < Tx, x > | = \| T \|$.