Math 621 Topology-II

Homework # 3 Due: April 1st 2012

Let R_l be the set of real numbers with the lower limit topology, R_u the set of real numbers with the usual topology, and R_d is the set of real numbers with the discrete topology.

1. Show that $R \times R$ with the dictionary order topology is homeomorphic to $R_d \times R_u$.
2. Describe the subspace topology on a line L in the plane topologized as $R_l \times R_u$ and also as $R_l \times R_l$.
3. Let $A \subseteq \overline{X}$, $f : A \to Y$ be a continuous function and Y be Housdorff space. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$ then g unique.
4. Show that $R \times R$ in the dictionary order topology is matrizable.
5. Let R^∞ denotes the subset of R^w (countable product) consisting of sequences $(x_1, x_2, ..., x_n, ...)$ which are eventually zero i.e., $x_i \neq 0$ for at most finitely many i. What is $\overline{R}^\infty$ in the box and product topologies on R^w?
6. Consider R^w with the box and product topologies.

   (a) In which topology (topologies) are $f, g, h : R \to R^w$ continuous?

   $$f(t) = (t, 2t, 3t, ...)$$
   $$g(t) = (t, t, t, ...)$$
   $$h(t) = \left(t, \frac{1}{2}t, \frac{1}{3}t, ... \right)$$

   (b) In which do the following sequences converge?

   $$w_1 = (1,1,1,1,...) \qquad x_1 = (1,1,1,1,...)$$
   $$w_2 = (0,2,2,2,...) \qquad x_2 = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ... \right)$$
   $$w_3 = (0,0,3,3,...) \qquad x_3 = \left(0, 0, \frac{1}{3}, \frac{1}{3}, ... \right)$$
   $$\vdots \qquad \vdots$$
   $$\langle w_n \rangle \qquad \langle w_n \rangle$$

$$
\begin{array}{ll}
y_1 = (1,0,0,0,\ldots) & z_1 = (1,1,0,0,\ldots) \\
y_2 = (\frac{1}{2},\frac{1}{2},0,0,\ldots) & z_2 = \left(\frac{1}{2},\frac{1}{2},0,0\ldots\right) \\
y_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0,\ldots) & z_3 = \left(\frac{1}{3},\frac{1}{3},0,0,\ldots\right) \\
\vdots & \vdots \\
\langle y_n \rangle & \langle z_n \rangle
\end{array}
$$

7. Let $f_n:[0,1]\to R$ be defined by $f_n(x)=x^n$ for each positive integer n. Show that $< f_n(x) >$ Converges for each x in $[0,1]$ but $< f_n >$ does not converge uniformly.

8. Let X be a topological space and Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions and let $< x_n > \overset{conges}{\to} x$ in X. If $< f_n > \overset{conges}{\to} f$ uniformly, show that $< f_n(x_n) > \overset{conges}{\to} f(x)$.

9. (a) Is R_l connected?
   (c) Show that R^w is not connected in the box topology.
   (d) Show that R^n and R are not homeomorphic if $n>1$.

10. Cantor's middle third set C.

$$A_0 = [0,1],\ A_1 = A_0 - \left(\frac{1}{2},\frac{2}{3}\right), A_2 = A_1 - \left[\left(\frac{1}{9},\frac{2}{9}\right)\cup\left(\frac{7}{9},\frac{8}{9}\right)\right]$$

$$A_n = A_{n-1} - \bigcup_{k=0}^{n-1}\left(\frac{1+3^k}{3^n},\frac{2+3^k}{3^n}\right)$$

$C = \bigcap_{n=1}^{\infty} A_n$ Is called "the" Cantor set (which is a subspace of $[0,1]$.

(a) C Is totally disconnected, i.e., a point is the largest connected subset.
(b) C Is compact
(c) $A_n =$ Union of finitely many closed intervals of length $\frac{1}{3^n}$. Show also that their endpoints are also in C.
(d) Show that every point of C is a limit point of C.
(e) C is uncountable.