Math 101 Quiz 4 Date 07/11/2012

Ex1: Find the slope of the tangent line to the curve $y = 2\sin(\pi x - y)$ at $(1,0)$

Solution: Using implicit differentiation, we have:

$$\frac{dy}{dx} = 2 \cdot \frac{d}{dx}(\pi x - y) \cdot \cos(\pi x - y)$$
$$= 2\left(\pi - \frac{dy}{dx}\right)\cos(\pi x - y).$$

Solving for $\frac{dy}{dx}$ implies that:

$$\frac{dy}{dx}\left(1 + 2\cos(\pi x - y)\right) = 2\pi\cos(\pi x - y)$$

i.e $\displaystyle \frac{dy}{dx} = \frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}$. Therefore the slope of the tangent line at $(1,0)$ is: $\displaystyle \left.\frac{dy}{dx}\right|_{\substack{x=1\\ y=0}} = \frac{-2\pi}{-1} = 2\pi$

Ex2 Find the normal line to the curve $x^2 + xy - y^2 = 1$ at $(2,3)$.

Solution: By implicit differentiation, we have:

$$\frac{d}{dx}\left(x^2 + xy - y^2\right) = \frac{d}{dx}1 \text{ . So:}$$

$2x + y + xy' - 2y'y = 0$. Thus $\displaystyle y' = \frac{2x+y}{2y+x}$.

The slope of the tangent line at $(2,3)$ is: $\displaystyle \left.\frac{2x+y}{2y+x}\right|_{\substack{x=2\\ y=3}} = \frac{7}{8}$

The normal line is perpendicular to the tangent line at $(2,3)$. Thus the slope of the normal line is $-\frac{8}{7}$.
Therefore an equation of the normal line is:

$$y = -\frac{8}{7}(x-2)+3 = -\frac{8}{7}x + \frac{37}{7}.$$

__Ex 3__: Let $y = \cos(xe^{x^2})$. Find $y''(0)$.

__Solution__: By the Chain rule, we have: $y' = -\frac{d}{dx}(xe^{x^2})\sin(xe^{x^2})$.

Now $\frac{d}{dx}(xe^{x^2}) = e^{x^2} + 2x^2e^{x^2}$.

Thus $y' = -\left(e^{x^2} + 2x^2e^{x^2}\right)\sin(xe^{x^2})$. So:

$$y'' = -\left(2xe^{x^2} + 4xe^{x^2} + 4x^3e^{x^2}\right)\sin(xe^{x^2})$$
$$-\left(e^{x^2} + 2x^2e^{x^2}\right)^2\cos(xe^{x^2}).$$

Therefore:

$$y''(0) = -1$$