KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICS AND STATISTICS

S121
MAT# 201
MAJOR EXAM I
Wednesday October 03, 2012
Time: 8:30 PM-10:30 PM
Location: Bldg 44

NAME: .................................................... ID: ......................... SECTI..ON: .............

Instructions: Formula sheet, calculator and mobile are not allowed.

SHOW ALL YOUR WORK

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Exercise #1: (12 pts) A curve \( C \) is defined by the parametric equations
\[
x = 1 - t^2, \quad y = 1 + t^2, \quad t \in [0, 3]
\]
(a) Find the point(s), if any, at which the normal line to the parametric curve \( C \) has slope 1.
(b) Determine where the curve is concave upward or downward.

(a) Since the slope of the normal line at a point is \( m_N = 1 \neq 0 \), the slope of the tangent line to the curve at the same point is \( m_T = \frac{-1}{m_N} = -1 \).
We have \( \frac{dx}{dt} = -3t \), \( \frac{dy}{dt} = 2t \), so that,
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{-3t} = -\frac{2}{3}, \quad \text{giving,}
\]
\[
m_T = \frac{dy}{dx} = -\frac{2}{3} \quad \text{and the point is}
\]
\[
(x, y) = \left(1 - \left(\frac{2}{3}\right)^3, 1 + \frac{1}{2} - \left(\frac{1}{3}\right)^2\right) = \left(\frac{26}{27}, \frac{11}{9}\right).
\]

(b) We have
\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dt}{dx} = \frac{d}{dt} \left( \frac{-2t}{-3t} \right)
\]
\[
= \left(\frac{1}{-3t^2}\right) \left[ -2 \left( -3t^2 \right) - \left(-6t\right)(-2t) \right] = \frac{6t - 6t}{27 \cdot \frac{2t-1}{9 \cdot 3}}
\]
\[
= \frac{6t - 6t}{27 \cdot \frac{2t-1}{9 \cdot 3}} = 2 \frac{t-1}{27}
\]
\[
\frac{d^2y}{dx^2} = \frac{1}{27} \quad \text{at} \quad t = 1
\]

So \( \frac{d^2y}{dx^2} \) changes
\[
\begin{array}{c|cc}
\text{Curve} & 0 & 1 & 2 \\
\hline
\text{Concave down} & + & - & +
\end{array}
\]
Exercise #2: (10 pts) Find the length of the curve

\[ x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \]

The length of the curve is given by

\[ L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

But \[ \frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t \]

and \[ \frac{dy}{dt} = \cos t - \cos t + \sin t = t \sin t \]

\[ L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} \, dt \]

\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t \, dt \]

\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi^2}{4} \]

\[ = \frac{\pi^2}{4} \]
Exercise #3 (10 pts) Write the polar equation $r = 2 \cos \theta + 2 \sin \theta$ in cartesian coordinates, then describe and sketch the graph of the resulting equation (including the axes!).

Multiplication of both sides of the polar equation by $r$ gives $r^2 = 2 r \cos \theta + 2 r \sin \theta$ but $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$, so

$$x^2 + y^2 = 2x + 2y$$

completing the squares yields

$$x^2 - 2x + y^2 - 2y = 0$$

which is an equation for the circle centered at $(1, 1)$ with radius $\sqrt{2}$. Note that the circle passes through the origin $(0, 0)$, $(0, 2)$, $(2, 0)$.
Exercise #4 (15 pts)

(a) Sketch the curves $C_1: r = 2 \cos 2\theta$ and $C_2: r = 1$ on the same axes. (b) Find the area inside the curve $C_1$ and outside the curve $C_2$ when

$\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

\[\text{Diagram showing the curves and axes with marked values of } \theta.\]

\[\text{Using the symmetries, we get the four-leaved rose as seen above.}\]

1. $\theta = \frac{\pi}{4}$

2. Intersection points:

\[2 \cos 2\theta = 1 \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{6}\]

So the area (of the shaded region) is

\[A = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( (2 \cos 2\theta)^2 - 1 \right) d\theta\]

\[= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( 4 \cos^2 2\theta - 1 \right) d\theta = \int_{0}^{\frac{\pi}{4}} \left( 4 \frac{1 + \cos 4\theta}{2} - 1 \right) d\theta\]

\[= \int_{0}^{\frac{\pi}{4}} \left( 2 \cos 4\theta + 1 \right) d\theta = 2 \cdot \frac{\sin 4\theta}{4} + \theta \bigg|_{0}^{\frac{\pi}{4}} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{2} = \pi\]
Exercise #5 (15pts)

(a) Graph \( r = \frac{1}{2} + \sin \theta \), \( \theta \in [0, 2\pi] \) showing all your steps.

(b) Find the slope of the tangent line to the graph at \( \theta = 0 \).

(a) The variations of \( r \) as a function of \( \theta \)
are as follows:

\[
\begin{array}{cccc}
\theta & 0 & \frac{\pi}{6} & \frac{\pi}{3} & \frac{\pi}{2} \\
r & 1 & \frac{3}{2} & \sqrt{3} & \frac{1}{2} \\
\end{array}
\]

\( \theta = \frac{7\pi}{6} \) and \( \theta = \frac{11\pi}{6} \) have been obtained as values of \( \theta \) for which \( r = 0 \) (r change in sign):

\[ \frac{1}{2} + \sin \theta = 0 \]

\[ \sin \theta = -\frac{1}{2} \quad \frac{\pi}{6} < \theta < \frac{7\pi}{6} \] or \( \frac{11\pi}{6} < \theta < \frac{13\pi}{6} \)

and \( \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \)

(b) The slope of the tangent line to the graph at \( \theta = 0 \) is:

\[
\frac{dy}{dx} \bigg|_{\theta=0} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \bigg|_{\theta=0} = \frac{\frac{1}{2}\left(1 + \sin \theta\right) \cos \theta}{\frac{1}{2}\left(1 + \sin \theta\right) \cos \theta} \bigg|_{\theta=0} = \frac{1}{2}.
\]
Exercise #6: (Spie) Find an equation of the sphere which has one of its diameters having end points (2, 1, 6) and (4, 3, 10).

The center of the sphere is at

\[ (\frac{2+4}{2}, \frac{1+3}{2}, \frac{6+10}{2}) = (3, 2, 8). \]

The diameter of the sphere is

\[ \sqrt{(4-2)^2 + (3-1)^2 + (10-6)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}. \]

and its radius is \( \frac{2\sqrt{6}}{2} = \sqrt{6} \). So an equation for the sphere is

\[ (x-3)^2 + (y-2)^2 + (z-8)^2 = 6. \]

Remark: It is worth noting that any point \( P(x, y, z) \) on the sphere with diameter \( AB \) where \( A(2, 1, 6) \), \( B(4, 3, 10) \) will satisfy \( \overrightarrow{AP} \cdot \overrightarrow{BP} = 0 \) so that

\[ (x-2)(x-4) + (y-1)(y-3) + (z-6)(z-10) = 0 \]

constitutes an equation of the sphere, though it is not in standard form.
Exercise #7: (8pts) Find a formula for the distance between the points with polar coordinates \((r_1, \theta_1)\) and \((r_2, \theta_2)\) where \(r_1, r_2 > 0\) and \(0 \leq \theta_1, \theta_2 < 2\pi\).

Let \(P_1 (r_1, \theta_1)\) and \(P_2 (r_2, \theta_2)\) in polar coordinates. The Cartesian coordinates would be

\[ P_1 (r_1 \cos \theta_1, r_1 \sin \theta_1) \quad \text{and} \quad P_2 (r_2 \cos \theta_2, r_2 \sin \theta_2) \]

giving the distance

\[
d = \sqrt{(r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2}
\]
Exercise #8 (12pts) Given $\vec{a} = 1 \hat{i} + 2 \hat{j} + 3 \hat{k}$ and $\vec{b} = 3 \hat{i} + 6 \hat{j} - 2 \hat{k}$

(a) Find the angle between $\vec{a}$ and $\vec{b}$.

\[
\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{(1)(3) + (2)(6) + (3)(-2)}{\sqrt{1^2 + 2^2 + 3^2} \sqrt{3^2 + 6^2 + (-2)^2}}
\]

\[
= \frac{3 + 12 - 6}{\sqrt{14} \sqrt{49}} = \frac{9}{7\sqrt{14}} \Rightarrow \theta = \cos^{-1}\left(\frac{9}{7\sqrt{14}}\right)
\]

(b) Find the vector projection of $\vec{a}$ onto $\vec{b}$.

\[
\text{proj}_b \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \frac{9}{7\sqrt{14}} \vec{b}
\]

(c) Find the scalar projection of $\vec{a}$ onto $\vec{b}$.

\[
\text{comp}_b \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(1)(3) + (2)(6) + (3)(-2)}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{9}{7}
\]
Exercise #9: (10pts) Determine, using only the dot product, whether the triangle with vertices $P(-1, 2, 3), Q(2, -2, 0)$ and $R(3, 1, -4)$ is a right angle triangle.

We have $\overrightarrow{PQ} = \langle 3, -5, -3 \rangle$ and $\overrightarrow{QR} = \langle 1, 3, -9 \rangle$ and $\overrightarrow{PR} = \langle 4, -1, -7 \rangle$. Thus,

$\overrightarrow{PQ} \cdot \overrightarrow{QR} = (3)(1) + (-5)(3) + (-3)(-9) = 3 - 15 + 27 = 15 \neq 0$ (2)

$\overrightarrow{QR} \cdot \overrightarrow{PR} = (1)(4) + (3)(-1) + (-9)(-7) = 4 - 3 + 63 = 64 \neq 0$ (2)

$\overrightarrow{PR} \cdot \overrightarrow{PQ} = (4)(3) + (-1)(-5) + (-7)(-3) = 12 + 5 + 21 = 37 \neq 0$ (2)

so none of the three angles of the triangle $PQR$ is a right angle. Hence, $PQR$ is not a right angle triangle.