Instructions: Formula sheets, calculators and mobiles are not allowed.

JUSTIFY ALL YOUR ANSWERS

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Mark
Exercise #1: (10 pts) Find the distance from the point D(1, 1, -3) to the plane containing the points A(1, 0, -2), B(0, -1, 2), C(1, 1, 1).

Equation of the plane containing the points A, B, C:
\[ \overrightarrow{AB} = \langle -1, -1, 4 \rangle, \quad \overrightarrow{AC} = \langle 0, 1, 3 \rangle \]

Normal vector:
\[ \overrightarrow{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 4 \\ 0 & 1 & 3 \end{vmatrix} = -7\mathbf{i} + 3\mathbf{j} - \mathbf{k} \]

Equation of the plane: \[ -7(x - 1) + 3(y - 0) - (z + 3) = 0 \]

That is: \[ -7x + 3y - z + 3 = 0 \]

Distance from D(1, 1, -3) to the plane ABC:
\[ d = \frac{|-7(1) + 3(1) - (-3) + 3|}{\sqrt{(-7)^2 + (3)^2 + (-1)^2}} = \frac{7}{\sqrt{59}} \]
Exercise #2: (11 pts) Find the area of the triangle whose vertices are P(1,3,2), Q(4,8,1) and R(2,2,3).

**Solution:**

we have \[ \overrightarrow{PQ} = (4-1)i + (8-3)j + (1-2)k \]
\[ = 3i - 5j - k \]

\[ \overrightarrow{PR} = (2-1)i + (2-3)j + (3-2)k \]
\[ = i - j + k \]

so that

\[ \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 3 & -5 & -1 \\ 1 & -1 & 1 \end{vmatrix} \]
\[ = (1-1)k - (3+5)j + (3+5)i 
\[ = 4i - 8j - 8k \]

The area of the triangle is therefore,

\[ A = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| \]
\[ = \frac{1}{2} \sqrt{(4)^2 + (-8)^2 + (-8)^2} \]
\[ = \frac{1}{2} \sqrt{16 + 64 + 64} \]
\[ = \frac{1}{2} \sqrt{144} \]
\[ = 2 \sqrt{6} \text{ unit}^2 \]
Exercise #3 (11 pts) Find an equation of the ellipsoid whose center is at $(1,1,1)$ and which passes through the points $(0,0,1),(1,1,\sqrt{5}+1)$ and $(1,0,3)$.

Solution: An equation of the ellipsoid is given by

$$a(x-1)^2 + b(y-1)^2 + c(z-1)^2 = 1$$

since it passes through $(0,0,1)$ we have,

$$a + b + c = 1$$

since it passes through $(1,1,\sqrt{5}+1)$ we have,

$$5c = 1$$

since it passes through $(1,0,3)$ we have,

$$b + 4c = 1$$

Thus, $a$, $b$, $c$ must satisfy the system,

$$\begin{cases} a + b + c = 1 \\ 5c = 1 \\ b + 4c = 1 \end{cases}$$

whose solution is $a = \frac{4}{5}$, $b = \frac{1}{5}$, $c = \frac{1}{5}$.

Therefore an equation of the ellipsoid is

$$\frac{4}{5}(x-1)^2 + \frac{1}{5}(y-1)^2 + \frac{1}{5}(z-1)^2 = 1.$$
Exercise #4: (15 pts) Consider the function

\[ f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2} \ln(z - x^2 - y^2) \]

(a) Find the domain of \( f \)

We must have \( z - x^2 - y^2 > 0 \) and \( 1 - x^2 - y^2 - z^2 > 0 \)

which gives the domain:

\[ \text{Domain } \mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 : z > x^2 + y^2 \text{ and } x^2 + y^2 + z^2 \leq 1\} \]

(b) Identify and sketch the domain of \( f \).

The domain of \( f \) consists of all points inside the paraboloid \( z = x^2 + y^2 \) and inside or on the sphere \( x^2 + y^2 + z^2 = 1 \) that is

\[
\begin{cases}
2 + y^2 = \frac{1 + \sqrt{5}}{2} \\
z = \frac{-1 + \sqrt{5}}{2} \\
x^2 + y^2 = \frac{1 + \sqrt{5}}{2}
\end{cases}
\]

Point \( A \) was obtained by solving \( x^2 + y^2 = 1 \). Points on the paraboloid are excluded.

Points on the sphere are included.
Exercise #4: (14 pts)
(a) Show that \( f(x, y) = \frac{x}{x+y} \) is differentiable at the point \((2, 1)\).

1. \( f_x(x, y) = \frac{y^2}{(x+y)^2} \) and \( f_y(x, y) = \frac{xy}{(x+y)^2} \)
2. \( f_x(2, 1) = \frac{1}{3} \) and \( f_y(2, 1) = \frac{1}{3} \)

(b) Find the linearization \( L(x, y, z) \) of \( f(x, y, z) = \tan^{-1}(xyz) \) at the point \((1, 1, 1)\).

1. \( f \) is differentiable at \((1, 1, 1)\) as composition of differentiable functions.
2. We have \( f(1, 1, 1) = \frac{\pi}{4} \).
3. \( \frac{\partial f}{\partial x}(1, 1, 1) = \frac{1}{2} \) and \( \frac{\partial f}{\partial y}(1, 1, 1) = \frac{1}{2} \) and \( \frac{\partial f}{\partial z}(1, 1, 1) = \frac{1}{2} \)

Thus, the linearization is

\[ L(x, y, z) = \left(1, 1, 1\right) + f_x(1, 1, 1) (x-1) + f_y(1, 1, 1) (y-1) + f_z(1, 1, 1) (z-1) \]

4. \[ L(x, y, z) = \frac{\pi}{4} + \frac{1}{2} (x-1) + \frac{1}{2} (y-1) + \frac{1}{2} (z-1) = \frac{\pi - 2}{4} + \frac{x + y + z}{2} \]
Exercise #5: (14 pts) Find the following limits if they exist,

(a) \[ \lim_{(x,y) \to (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} \]

Using polar coordinates we have \( x = r \cos \theta, \)
\( y = r \sin \theta \) so that
\[ \lim_{(x,y) \to (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} = \lim_{r \to 0} \frac{e^{-r^2} - 1}{r^2} = \lim_{r \to 0} \frac{e^{-r^2}}{r^2} \]
\[ = \lim_{r \to 0} \left( -\frac{e^{-r^2}}{2} \right) = \lim_{r \to 0} \frac{-e^{-r^2}}{2} = -1. \]

(b) \[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} \]

Along the path \( y = 0 \) we have
\[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} = \lim_{x \to 0} \frac{0}{x^2} = 0. \]

Along the path \( x = y^3 \) we have
\[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} = \lim_{y \to 0} \frac{y^3}{y^6 + y^6} = \lim_{y \to 0} \frac{y^3}{2y^6} = \lim_{y \to 0} \frac{1}{2y^3} = \frac{1}{2}. \]

These two limits are different.

(1) The limit does not exist.
(2) Other paths are possible.)
Exercise #7: (11 pts) The temperature of a body at a point \((x, y, z)\) is given by
\[ T = e^{xy} - xy^2 - x^2yz. \]

Determine the direction of the greatest drop in the temperature at the point \((1, -1, 2)\).

\[
\text{solution:} \quad \text{The greatest drop (decrease) at} \ \text{the point} \ \( (1, -1, 2) \) \ \text{is in the direction of the negative gradient at that point.}
\]

\[
\nabla T = (e^{xy} - y^2 - 2xy)\mathbf{i} + (xe^y - 2xy - x^2y)\mathbf{j} + (xe^y - 2xy - x^2y)\mathbf{k}
\]

\[
(\nabla T)(1, -1, 2) = \langle e^0 - 3, -e^0, -1 \rangle
\]

\[
\text{is the direction of the greatest drop in the temperature at the point} \ \( (1, -1, 2) \).
Exercise #8 (14 pts) Identify the following surfaces and sketch each one in 3D-space,

(a) \( x^2 + 4y + 9z^2 = 0 \)

\[ z = \frac{1}{4} x^2 + \frac{9}{4} z^2 = \left( \frac{x}{2} \right)^2 + \left( \frac{z}{\frac{3}{2}} \right)^2 \]

Elliptic paraboloid along the y-axis in the direction \( y < 0 \).

(b) \( x^2 - 4y^2 + z^2 - 6x - 8y - 2z + 6 = 0 \)

\[ x^2 - 6x + 9 - 4y^2 + \frac{z^2}{2} - 2z + 6 = 0 \]
\[ (x - 3)^2 - 9 - 4(y + 1)^2 + 4 + (\frac{z-1}{2})^2 + 6 = 0 \]
\[ (x - 3)^2 - (\frac{y+1}{2})^2 + (z-1)^2 = 0 \]

Cylinder with vertex at \((3,-1,1)\) and axis along the y-axis.