

Exercise #1: (12pts) A curve C is defined by the parametric equations

$$x = 1 - t^3, y = 1 + t - t^2, t \in [0, 3]$$

(a) Find the point(s), if any, at which the normal line to the parametric curve C has slope 1.

(b) Determine where the curve is concave upward or downward.

(a) Since the slope of the normal line at a point is $m_N = 1 \neq 0$, the slope of the tangent line to the curve at the same point is $m_T = -\frac{1}{m_N} = -1$.

We have $\frac{dx}{dt} = -3t^2$, $\frac{dy}{dt} = 1 - 2t$, so that,

$$m_T = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-2t}{-3t^2} = -1, \text{ giving,}$$

(2) $3t^2 + 2t - 1 = 0$ whose solutions are

$t = -1 \notin [0, 3]$ and $t = \frac{1}{3} \in [0, 3]$. Therefore, there is only one point at which the normal line to the parametric curve C has slope 1 and the point is

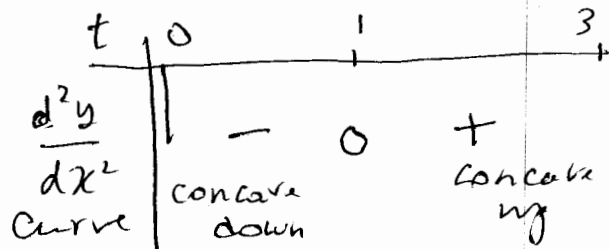
$$(2) (x, y) = \left(1 - \left(\frac{1}{3}\right)^3, 1 + \frac{1}{3} - \left(\frac{1}{3}\right)^2\right) = \left(\frac{26}{27}, \frac{11}{9}\right).$$

(b) We have

$$(2) \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left(\frac{1-2t}{-3t^2} \right)}{-3t^2} \quad (2)$$

$$= \frac{1}{(-3t^2)^3} \left[-2(-3t^2) - (-6t)(1-2t) \right] = \frac{6t^2 - 6t}{27t^6} = \frac{2t-1}{9t^5}$$

so



(2)

Exercise #2:(10pts) Find the length of the curve

$$x = \cos t + t \sin t, y = \sin t - t \cos t, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The length of the curve is given by

$$(3) \quad L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

but $\frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t$
(2) and $\frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t$

so $L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt$

$$(2) \quad = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{t^2(\cos^2 t + \sin^2 t)} dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t| dt = 2 \int_0^{\frac{\pi}{2}} t dt = t^2 \Big|_0^{\frac{\pi}{2}}$$

$$(1) \quad = \frac{\pi^2}{4} \quad (2)$$

Exercise #3: (10pts) Write the polar equation $r = 2 \cos \theta + 2 \sin \theta$ in cartesian coordinates, then describe and sketch the graph of the resulting equation (including the axes!).

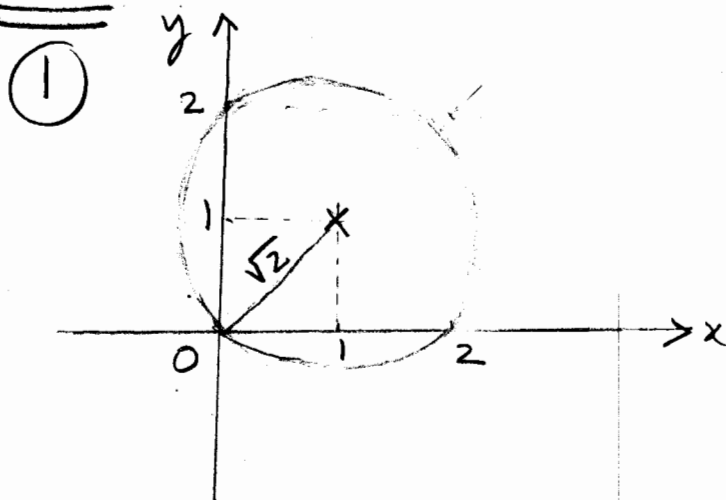
Multiplication of both sides of the polar equation by r gives $r^2 = 2r \cos \theta + 2r \sin \theta$ but $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$,
so $x^2 + y^2 = 2x + 2y$

$$x^2 - 2x + y^2 - 2y = 0$$

completing the squares yields

$$(x-1)^2 + (y-1)^2 = 2$$

which is an equation for the circle centered at $(1, 1)$ with radius $\sqrt{2}$. Note that the circle passes through the origin $(0, 0)$, $(0, 2)$, $(2, 0)$



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Exercise #4: (15pts)

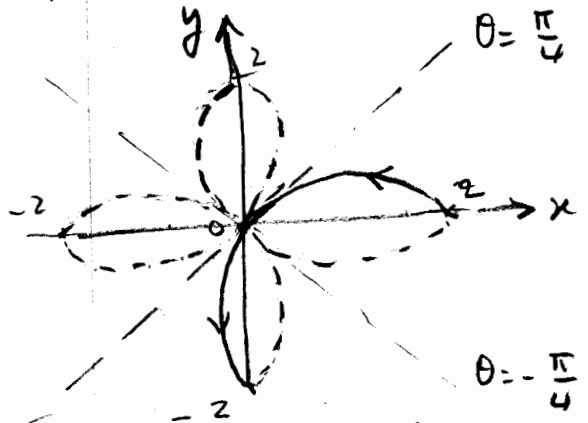
(a) Sketch the curves $C_1 : r = 2 \cos 2\theta$ and $C_2 : r = 1$ on the same axes. (b) Find the area inside the curve C_1 and outside the curve C_2 when $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

(a) $C_1 : r = 2 \cos 2\theta$

changing θ into $-\theta$ and θ into $\pi - \theta$ do not change the value of r so the curve is symmetric with respect to the x-axis and the y-axis.

(1)

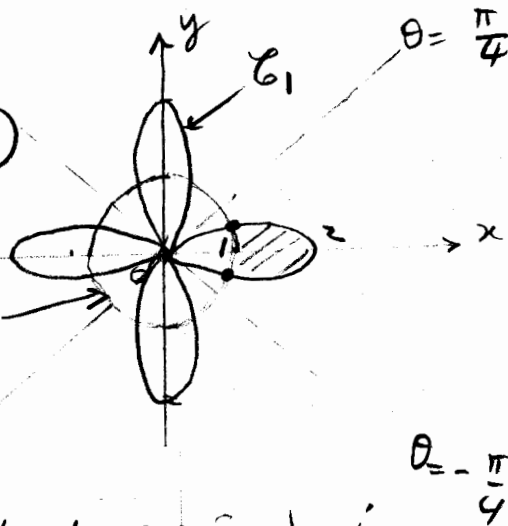
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
2θ	0	$\frac{\pi}{2}$	π
r	2	0	-2



Using the symmetries, we get the four-leafed rose as seen above.

On the same axes,

(5)



(b) Intersection points!

(2) $2 \cos 2\theta = 1 \therefore \cos 2\theta = \frac{1}{2} C_2$
 $\therefore 2\theta = \pm \frac{\pi}{3}$ that is
 $\theta = \pm \frac{\pi}{6}$

So the area (of the hatched region) is

(2) $A = \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} [(2 \cos 2\theta)^2 - (1)^2] d\theta$
 $= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} [4 \cos^2 2\theta - 1] d\theta = \int_0^{\frac{\pi}{6}} [4 \frac{\cos 4\theta + 1}{2} - 1] d\theta$
 $= \int_0^{\frac{\pi}{6}} [2 \cos 4\theta + 1] d\theta = 2 \frac{\sin 4\theta}{4} + \theta \Big|_0^{\frac{\pi}{6}} = \frac{1}{2} \sin \frac{2\pi}{3} + \frac{\pi}{6} = \frac{\sqrt{3}}{4} + \frac{\pi}{6}$

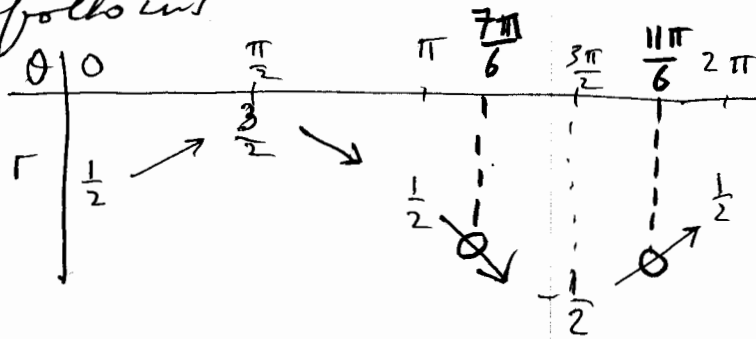
(2) (1)

Exercise #5: (15pts)

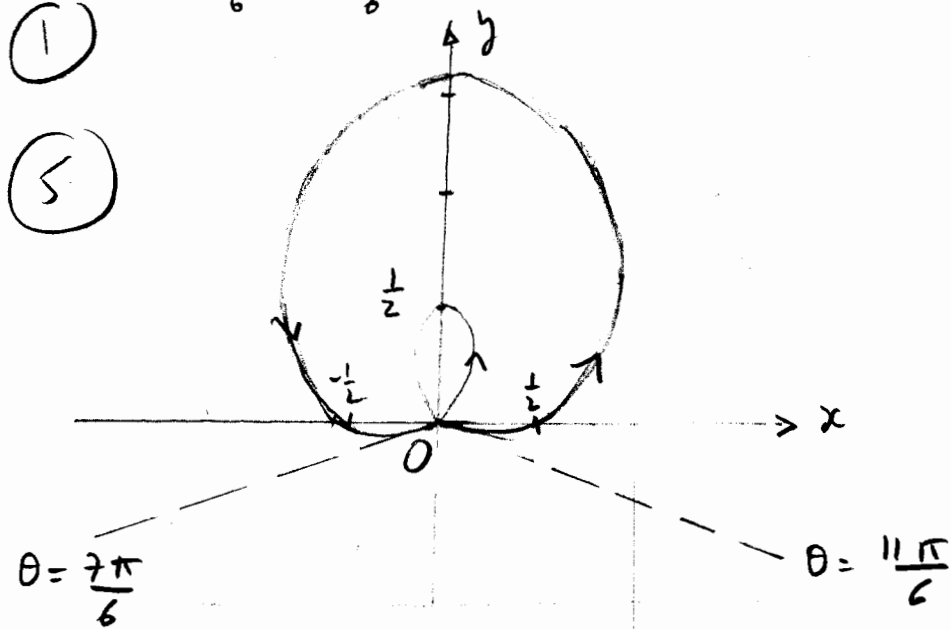
- (a) Graph $r = \frac{1}{2} + \sin \theta$, $\theta \in [0, 2\pi)$ showing all your steps.
 (b) Find the slope of the tangent line to the graph at $\theta = 0$.

(a) The variations of r as a function of θ are as follows

(2)



$\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$ have been obtained as values of θ for which $r = 0$ (r changes sign) : $\frac{1}{2} + \sin \theta = 0 \therefore \sin \theta = -\frac{1}{2} \therefore \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ (1)
 and $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ (1)



(b) The slope of the tangent line to the graph at $\theta = 0$ is

$$\frac{dy}{dx} \Big|_{\theta=0} = \frac{dy/d\theta}{dx/d\theta} \Big|_{\theta=0} = \frac{\frac{d}{d\theta} \left[\left(\frac{1}{2} + \sin \theta \right) \cos \theta \right]}{\frac{d}{d\theta} \left[\left(\frac{1}{2} + \sin \theta \right) \sin \theta \right]} \Big|_{\theta=0} = \frac{\cos \theta \cos \theta + \left(\frac{1}{2} + \sin \theta \right) (-\sin \theta)}{\cos^2 \theta - \left(\frac{1}{2} + \sin \theta \right) \cos \theta} \Big|_{\theta=0} = \frac{1}{2}$$

(2) (2) (1)

Exercise #6:(8pts) Find an equation of the sphere which has one of its diameters having end points $(2, 1, 6)$ and $(4, 3, 10)$.

The center of the sphere is at

$$\textcircled{2} \left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{6+10}{2} \right) = (3, 2, 8).$$

The diameter of the sphere is

$$\textcircled{2} \sqrt{(4-2)^2 + (3-1)^2 + (10-6)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

$\textcircled{1}$ and its radius is $\frac{2\sqrt{6}}{2} = \sqrt{6}$. So an equation for the sphere is

$$\textcircled{3} (x-3)^2 + (y-2)^2 + (z-8)^2 = 6.$$

Remark: It is worth noting that any point $P(x, y, z)$ on the sphere with diameter AB where $A(2, 1, 6)$, $B(4, 3, 10)$ will satisfy $\vec{AP} \cdot \vec{BP} = 0$ so that

$$(x-2)(x-4) + (y-1)(y-3) + (z-6)(z-10) = 0$$

constitutes an equation of the sphere, though it is not in standard form.

Exercise #7: (8pts) Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2) where $r_1, r_2 > 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$.

Let $P_1 (r_1, \theta_1)$ and $P_2 (r_2, \theta_2)$
in polar coordinates. The cartesian
coordinates would be

$P_1 (r_1 \cos \theta_1, r_1 \sin \theta_1)$ and $P_2 (r_2 \cos \theta_2, r_2 \sin \theta_2)$
giving the distance

$$d = \sqrt{(r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2}$$

Exercise #8: (12pts) Given $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} + 6\vec{j} - 2\vec{k}$

(a) Find the angle between \vec{a} and \vec{b} .

(b) Find the vector projection of \vec{a} onto \vec{b} .

(c) Find the scalar projection of \vec{a} onto \vec{b} .

$$(a) \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{(1)(3) + (2)(6) + (3)(-2)}{\sqrt{1^2 + 2^2 + 3^2} \sqrt{3^2 + 6^2 + (-2)^2}}$$
$$= \frac{3 + 12 - 6}{\sqrt{14} \sqrt{49}} = \frac{9}{7\sqrt{14}} \quad \therefore \theta = \cos^{-1} \left(\frac{9}{7\sqrt{14}} \right)$$

$$(b) \text{Proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$
$$= \frac{(1)(3) + (2)(6) + (3)(-2)}{(3^2 + 6^2 + (-2)^2)} (3\vec{i} + 6\vec{j} - 2\vec{k})$$
$$= \frac{9}{49} (3\vec{i} + 6\vec{j} - 2\vec{k}) = \frac{27}{49} \vec{i} + \frac{54}{49} \vec{j} - \frac{18}{49} \vec{k}$$

$$(c) \text{Comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(1)(3) + (2)(6) + (3)(-2)}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{9}{7}$$

Exercise #9:(10pts) Determine, using only the dot product, whether the triangle with vertices $P(-1, 2, 3)$, $Q(2, -2, 0)$ and $R(3, 1, -4)$ is a right angle triangle.

We have $\vec{PQ} = \langle 3, -4, -3 \rangle$, $\vec{QR} = \langle 1, 3, -4 \rangle$ and $\vec{PR} = \langle 4, -1, -7 \rangle$. Thus,

$$\vec{PQ} \cdot \vec{QR} = (3)(1) + (-4)(3) + (-3)(-4) = 3 - 12 + 12 = 3 \neq 0$$

$$\vec{QR} \cdot \vec{PR} = (1)(4) + (3)(-1) + (-4)(-7) = 4 - 3 + 28 = 29 \neq 0$$

$$\vec{PR} \cdot \vec{PQ} = (4)(3) + (-1)(-4) + (-7)(-3) = 12 + 4 + 21 = 37 \neq 0$$

So none of the three angles of the triangle PQR is a right angle. Hence, PQR is not a right angle triangle.

(2)