Instructions: Formula sheets, calculators and mobiles are not allowed.

**JUSTIFY ALL YOURS ANSWERS**

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Exercise #1: (10 pts) Find the distance from the point \( D(1, 1, -3) \) to the plane containing the points \( A(1, 0, -2) \), \( B(0, -1, 2) \), \( C(1, 1, 1) \).

Section 1:

Equation of the plane containing the points \( A, B, C \):

\[ \overrightarrow{AB} = \langle -1, -1, 4 \rangle, \quad \overrightarrow{AC} = \langle 0, 1, 3 \rangle \]

Normal vector:

\[ \overrightarrow{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & -1 & 4 \\ 0 & 1 & 3 \end{vmatrix} = 7i + 3j - k \]

Equation of the plane \( \langle A, B, C \rangle \):

\[ -7(x - 1) + 3(y + 2) - (z + 5) = 0 \]

That is:

\[-7x + 7 + 3y + 6 - z - 5 = 0 \]

Distance from \( D(1, 1, -3) \) to the plane \( \overrightarrow{ABC} \):

\[ d = \frac{|-7(1) + 3(1) - (-3) + 5|}{\sqrt{(-7)^2 + (3)^2 + (-1)^2}} = \frac{\sqrt{59}}{\sqrt{59}} \]
Exercise #2: (11 pts) Find the area of the triangle whose vertices are P(1, 3, 2), Q(4, 8, 1) and R(2, 2, 3).

**Solution:**

we have

\[
\overrightarrow{PQ} = (4-1)i + (8-3)j + (1-2)k
\]
\[
= 3i + 5j - k
\]

\[
\overrightarrow{PR} = (2-1)i + (2-3)j + (3-2)k
\]
\[
= i - j + k
\]

so that

\[
\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix}
i & j & k \\
3 & 5 & -1 \\
1 & -1 & 1 \\
\end{vmatrix}
\]
\[
= (5-1)i - (3-(-1))j + (-1-5)k
\]
\[
= 4i - 4j - 6k
\]

the area of the triangle is therefore,

\[
A = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|
\]
\[
= \frac{1}{2} \sqrt{(4)^2 + (-4)^2 + (-6)^2}
\]
\[
= \frac{1}{2} \sqrt{16 + 16 + 36}
\]
\[
= \frac{1}{2} \sqrt{68}
\]
\[
= \frac{1}{2} \sqrt{96}
\]
\[
= 2\sqrt{6} \quad \text{unit}^2
\]
Exercise #3: (11 pts) Find an equation of the ellipsoid whose center is at \((1, 1, 1)\) and which passes through the points \((0, 0, 1), (1, 1, \sqrt{5} + 1)\) and \((1, 0, 3)\).

Solution:

An equation of the ellipsoid is given by

\[
a (x - 1)^2 + b (y - 1)^2 + c (z - 1)^2 = 1
\]

Since it passes through \((1, 0, 3)\) we have,

\[
a + b = 1
\]

Since it passes through \((1, 1, \sqrt{5} + 1)\) we have,

\[
s c = 1
\]

Since it passes through \((1, 0, 3)\) we have,

\[
b + 4 c = 1
\]

Thus, \(a, b, c\) must satisfy the system,

\[
\begin{align*}
a + b &= 1 \\
c &= 1 \\
b + 4 c &= 1
\end{align*}
\]

whose solution is \(a = \frac{4}{5}, b = \frac{1}{5}, c = \frac{1}{5}\).

Therefore an equation of the ellipsoid is

\[
\frac{4}{5} (x - 1)^2 + \frac{1}{5} (y - 1)^2 + \frac{1}{5} (z - 1)^2 = 1.
\]
Exercise #4: (15 pts) Consider the function
\[ f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2 \ln(z - x^2 - y^2)} \]

(a) Find the domain of \( f \)

We must have \( z - x^2 - y^2 > 0 \) and \( 1 - x^2 - y^2 - z^2 \ln(z - x^2 - y^2) > 0 \), which gives the domain:
\[ \text{Domain } \mathcal{D} = \{ (x, y, z) \in \mathbb{R}^3 : z > x^2 + y^2 \text{ and } x^2 + y^2 + z^2 \leq 1 \} \]

(b) Identify and sketch the domain of \( f \).

The domain of \( f \) consists of all points inside the paraboloid \( z = x^2 + y^2 \) and inside or on the sphere \( x^2 + y^2 + z^2 = 1 \), that is:

1. Equation of the circle:
   \[ x^2 + y^2 = \frac{1 + \sqrt{5}}{2} \]
   \[ z = \frac{1 + \sqrt{5}}{2} \]

2. A was obtained by solving \( z + z^2 = 1 \). Points on the paraboloid are excluded. Points on the sphere are included.
Exercise #4: (14 pts)
(a) Show that \( f(x, y) = \frac{x+y}{x^2+y^2} \) is differentiable at the point (2, 1).

1. \( f_x(x, y) = \frac{(x+y)\cdot 2 - x \cdot 2}{(x^2+y^2)^2} = \frac{2y}{(x^2+y^2)^2} \)

2. \( f_y(x, y) = \frac{(x+y)\cdot 2 - y \cdot 2}{(x^2+y^2)^2} = \frac{2x}{(x^2+y^2)^2} \)

3. \( f \) is continuous and has continuous partial derivatives at \((2, 1)\) and in a small neighborhood of \((2, 1)\). Thus, \( f \) is differentiable at \((2, 1)\).

(b) Find the linearization \( L(x, y, z) \) of \( f(x, y, z) = \tan^{-1}(xyz) \) at the point \((1, 1, 1)\).

1. \( \tan^{-1}(1) = \frac{\pi}{4} \) so \( \tan^{-1}(1, 1, 1) = \frac{\pi}{4} \)

2. \( L_x(x, y, z) = \frac{y^2}{1+y^2} \Rightarrow L_x(1, 1, 1) = \frac{1}{2} \)

3. \( L_y(x, y, z) = \frac{x^2}{1+x^2} \Rightarrow L_y(1, 1, 1) = \frac{1}{2} \)

4. \( L_z(x, y, z) = \frac{xy}{1+x^2} \Rightarrow L_z(1, 1, 1) = \frac{1}{2} \)

Thus, the linearization is

5. \( L(x, y, z) = \left(1, 1, 1\right) + L_x(1, 1, 1)(x-1) + L_y(1, 1, 1)(y-1) + L_z(1, 1, 1)(z-1) \)

6. \( L(x, y, z) = \frac{\pi}{4} + \frac{1}{2} (x-1)^2 + (y-1)^2 + (z-1)^2 = \frac{\pi - 2}{4} + \frac{x^2+y^2+z^2}{2} \)
Exercise #5: (14 pts) Find the following limits if they exist,

(a) \[ \lim_{(x,y) \to (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} \]

Using polar coordinates \( x = r \cos \theta \), \( y = r \sin \theta \) we have \( x = r \cos \theta \), \( y = r \sin \theta \) and

\[ \lim_{(x,y) \to (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} = \lim_{r \to 0} \frac{e^{-r^2} - 1}{r^2} \]

\[ = \lim_{r \to 0} \frac{e^{-r^2} - 1}{r} = \lim_{r \to 0} \frac{e^{-r^2}}{r} = 1 \]

(b) \[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} \]

Along the path \( y = 0 \) we have

\[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} = \lim_{y \to 0} \frac{0}{x^2 + 0^2} = 0 \]

Along the path \( x = y^2 \) we have

\[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} = \lim_{y \to 0} \frac{y^2 y^3}{y^4 + y^2} = \lim_{y \to 0} \frac{y^5}{y^2 + y^2} = \lim_{y \to 0} \frac{y^2}{y^2} = 1 \]

1. These two limits are different if

2. \[ \lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^2} \] does not exist

(Other paths are possible.)
Exercise #7: (11 pts) The temperature of a body at a point \((x, y, z)\) is given by
\[ T = e^{xy} - xy^2 - x^2yz. \]

Determine the direction of the greatest drop in the temperature at the point \((1, -1, 2)\).

Solution: The greatest drop (decrease) at the point \((1, -1, 2)\) is in the direction of the negative gradient at that point. 

\[ \nabla T = \left( y e^{xy} - y^2 - 2xyz \right) \hat{i} + \left( x e^{xy} - 2xy - x^2z \right) \hat{j} + (-x^2y) \hat{k}, \]

so

\[ -\nabla T (1, -1, 2) = \langle -e^{-1}, -e^{-1}, -1 \rangle \]

is the direction of the greatest drop in the temperature at the point \((1, -1, 2)\).
Exercise #8 (14 pts) Identify the following surfaces and sketch each one in 3D-space,

(a) \( z^2 + 4y + 9z^2 = 0 \)

\[
-\frac{1}{4} x^2 + \frac{9}{4} y^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2
\]

Elliptic paraboloid along the y-axis in the direction \( y < 0 \).

(b) \( x^2 - 4y^2 + z^2 - 6x - 8y - 2z + 6 = 0 \)

\[
\begin{align*}
\frac{x^2}{6} + \frac{y^2}{1} + \frac{z^2}{2} &= -2x - 8y + 2z + 6 = 0 \\
[(x - 3)^2 - 9] - 4(y - 1)^2 + 4 - (z - 1)^2 + 6 &= 0 \\
(x - 3)^2 - \left(\frac{y+1}{2}\right)^2 + (z - 1)^2 &= 0
\end{align*}
\]

Circular cone with vertex at \((3, -1, 1)\) and axis along the y-axis.