REMARKS FOR INSTRUCTORS

1) For the MCQ: this is the master key so all (a) are solutions to the corresponding exercises EXCEPT

Exercise #2 solution is $-\pi$ that is (b) in master key (not $\pi$)

Exercise # 3
whose solution should be 10 or 6, which is not included in the 5 choices given and so is discarded. Every student will be given full mark for that exercise.

2) Exercise 11 on Lagrange multiplier there was a missprint $z$ instead of 2 which led to little computational problem but the problem is workable and the solution is given.

3) Exercise 13 we consider $0 \leq \theta \leq \frac{\pi}{4}$ otherwise there will be two integrals to evaluate.
PART I: MCQ

1. (7pts) The area enclosed by the ellipse \( \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \) where \( A, B > 0 \) is
   (a) \( \pi AB \)
   (b) \( AB \)
   (c) \( \frac{AB}{2} \)
   (d) \( 2AB \)
   (e) \( \frac{\pi}{2}(A^2 + B^2) \)

2. (7pts) The slope of the tangent line to the polar curve \( r = \frac{1}{\theta} \) at \( \theta = \pi \) is
   (a) \( \pi \)
   (b) \( -\pi \)
   (c) \( -\frac{1}{\pi^2} \)
   (d) \( \frac{1}{\pi} \)
   (e) 0
PART I: MCQ (cont’)

3. (7pts) If a sphere with center at O(3, 8, 1) passes through P(4, 3, −1) and Q(8, Y, 0), then Y is equal to
   (a) 4 or 8
   (b) 4
   (c) 12 or 6
   (d) 12
   (e) 6

PART I: MCQ (cont’)

4. (7pts) The angle between the vectors $u = \langle 3, 4, -1 \rangle$ and $v = \langle 2, -1, -3 \rangle$ is
   (a) $\cos^{-1}\left(\frac{5}{2\sqrt{14}}\right)$
   (b) $\cos^{-1}\left(\frac{4}{\sqrt{11}}\right)$
   (c) $\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)$
   (d) $\cos^{-1}\left(\frac{2}{\sqrt{11}}\right)$
   (e) $\cos^{-1}\left(\frac{5}{\sqrt{11}}\right)$
PART I: MCQ (cont’)

5. (7pts) The volume of the parallelepiped determined by the three vectors \( \mathbf{u} = \langle 3, 2, 1 \rangle, \mathbf{v} = \langle 1, 1, 2 \rangle \) and \( \mathbf{w} = \langle 1, 3, 3 \rangle \) is
   (a) 9
   (b) 13
   (c) 12
   (d) 10
   (e) 8

PART I: MCQ (cont’)

6. (7pts) An equation of the plane through \((-1, -2, 3)\) and perpendicular to both the planes \(x - 3y + 2z = 7\) and \(2x - 2y - z = -3\) is
   (a) \(7x + 5y + 4z + 5 = 0\)
   (b) \(7x + 5y + 3z + 8 = 0\)
   (c) \(5x + 7y + 4z + 7 = 0\)
   (d) \(5x + 7y + 3z + 10 = 0\)
   (e) \(4x + 3y + 7z - 11 = 0\)
PART I: MCQ (cont’)

7. (7pts) The distance of the point P(4, 1, −3) to the plane that passes through the points A(1, 1, −1), B(−2, 0, 1) and C(0, 1, 3) is
   (a) $\frac{10}{\sqrt{117}}$
   (b) $\frac{6}{\sqrt{119}}$
   (c) $\frac{3}{\sqrt{11}}$
   (d) $\frac{13}{\sqrt{11}}$
   (e) $\frac{14}{\sqrt{109}}$

PART I: MCQ (cont’)

8. (7pts) Let
   \[ f(x, y) = \frac{x^2y}{x^4 + y^2} \]
   then \( \lim_{x \to 0, y \to 0} f(x, y) \) does not exist since
   (a) \( \lim_{x \to 0} f(x, y) \) along the curve \( y = \alpha x^2 \) depends on \( \alpha \)
   (b) \( \lim_{x \to 0} f(x, y) \) along the curve \( y = \alpha x^2 \) is independent of \( \alpha \)
   (c) \( \lim_{x \to 0} f(x, y) \) along the curve \( y = x^2 \) is equal to \( \frac{1}{2} \)
   (d) \( \lim_{x \to 0} f(x, y) \) along the curve \( y = 0 \) is equal to 0
   (e) \( \lim_{y \to 0} f(x, y) \) along the curve \( x = 0 \) is equal to 0.
9. (7pts) If \( z = f(\frac{x}{y}) \) where \( f \) is a differentiable function, then \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \) is equal to

(a) \( y \frac{x}{y} f'\left(\frac{x}{y}\right) \)
(b) \( x f'\left(\frac{x}{y}\right) + \frac{x}{y} f'\left(\frac{x}{y}\right) \)
(c) \( \frac{x-y}{y} f'\left(\frac{x}{y}\right) \)
(d) 0
(e) \( y f'\left(\frac{x}{y}\right) - \frac{x}{y^2} f'\left(\frac{x}{y}\right) \)

10. (7pts) The linearization \( L(x, y, z) \) of \( f(x, y, z) = xyz^2 + y^2z + 2x \) at the point \((1, 0, 1)\) is

(a) \( L(x, y, z) = 2x + y \)
(b) \( L(x, y, z) = 2x + y - 2 \)
(c) \( L(x, y, z) = x + y + 2z + 1 \)
(d) \( L(x, y, z) = 2x + z + z + 2 \)
(e) \( L(x, y, z) = x + y + z \)
PART II: Written

11. (14pts) Find the minimum and maximum of \( f(x, y, z) = -x + 2y + 2z \)
subject to the constraints
\[
\begin{align*}
  g(x, y, z) &= x^2 + y^2 - z = 0 \\
  h(x, y, z) &= y + 2z - 1 = 0
\end{align*}
\]

Let \( F = f - \lambda g - \mu h \) then \( \nabla F = 0 \) gives the system
\[
\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial \lambda} = 0, \quad \frac{\partial F}{\partial \mu} = 0,
\]
that is
\[
\begin{align*}
  \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} + \mu \frac{\partial h}{\partial x} \\
  \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} + \mu \frac{\partial h}{\partial y} \\
  \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} + \mu \frac{\partial h}{\partial z} \\
  g &= 0 \\
  h &= 0
\end{align*}
\]
hence,
\[
\begin{align*}
  -1 &= 2\lambda x \\
  2 &= 2\lambda y + \mu \\
  2 &= -\lambda + 2\mu \\
  x^2 + y^2 - z &= 0 \\
  y + 2z - 1 &= 0
\end{align*}
\]
from which we get
\[
\begin{align*}
  x &= \frac{-1}{2\lambda} \\
  y &= \frac{2 - \mu}{2\lambda} \\
  \lambda &= 2\mu - 2
\end{align*}
\]
Replacing \( \lambda \) in terms of \( \mu \) in \( x \) and \( y \) we get
\[
\begin{align*}
  x &= \frac{-1}{4(\mu - 1)} \\
  y &= \frac{2 - \mu}{4(\mu - 1)}
\end{align*}
\]
From the constraints equations we have
\[ y + 2(x^2 + y^2) = 1 \]
giving,
\[ \frac{2 - \mu}{4(\mu - 1)} + 2 \left( \frac{1}{16(\mu - 1)^2} + \frac{(2 - \mu)^2}{16(\mu - 1)^2} \right) = 1 \]
Reducing to the same denominator and simplifying, we get
\[ 9\mu^2 - 18\mu + 7 = 0 \]
whose solutions are
\[ \mu_1 = \frac{1}{3}(3 - \sqrt{2}) \text{ and } \mu_2 = \frac{1}{3}(3 + \sqrt{2}). \]

Hence,
\[ \lambda_1 = -\frac{2\sqrt{2}}{3} \text{ and } \lambda_2 = \frac{2\sqrt{2}}{3} \]
and
\[ x_1 = \frac{3\sqrt{2}}{8} \text{ and } x_2 = -\frac{3\sqrt{2}}{8} \]
\[ y_1 = -\frac{1}{8}(2 + 3\sqrt{2}) \text{ and } y_2 = \frac{1}{8}(-2 + 3\sqrt{2}) \]
\[ z_1 = \frac{1}{16}(10 + 3\sqrt{2}) \text{ and } z_2 = \frac{1}{16}(10 - 3\sqrt{2}) \]

Evaluation of \( f \) at the points \( M_1(x_1, y_1, z_1) \) and \( M_2(x_2, y_2, z_2) \) gives \( f_1 \approx -0.31 \) and \( f_2 \approx 1.81 \) which turn out to be the minimum of \( f \) and the maximum of \( f \) respectively.
12. (14pts) Evaluate

\[ I = \int_0^8 \int_{y^{1/3}}^2 e^{x^4} \, dx \, dy \]

Since we cannot perform the inner integration, and the integrand is continuous, we shall reverse the order of integrations.

we have,

\[
I = \int_0^2 \int_0^{x^3} e^{x^4} \, dy \, dx \\
= \int_0^2 e^{x^4} \int_0^{x^3} dy \, dx \\
= \int_0^2 e^{x^4} x^3 \, dx \\
= \left( \frac{1}{4} e^{x^4} \right) \bigg|_{x=0}^{x=2} \\
= \frac{1}{4} (e^{16} - 1)
\]
PART II: Written (cont’)

13. (14pts) Integrate the function \( f(x, y) = xy \) over the region bounded by the four-leaved rose \( r = \cos 2\theta \) in the first quadrant \( (0 \leq \theta \leq \frac{\pi}{4}) \).

Note: HERE WE SHALL CONSIDER \( 0 \leq \theta \leq \frac{\pi}{4} \) OTHERWISE THERE WILL BE TWO INTEGRALS TO EVALUATE

\[
I = \int_{0}^{\frac{\pi}{4}} \left( \int_{0}^{r} r \cos \theta r \sin \theta r dr d\theta \right)
\]

\[
= \int_{0}^{\frac{\pi}{4}} \left[ \int_{0}^{\frac{r^4}{4}} r^2 \cos 2\theta \sin \theta d\theta \right] r dr d\theta
\]

\[
= \frac{1}{8} \int_{0}^{\frac{\pi}{4}} \cos^4 2\theta \sin 2\theta d\theta
\]

\[
= \left[ \frac{1}{8} \cos^5 2\theta \right]_{\theta=0}^{\frac{\pi}{4}}
\]

\[
= \frac{1}{80}
\]
14. (14 pts) Integrate the function \( f(x, y, z) = 6xy \) over the solid \( E \) that lies under the plane \( z = 1 + x + y \) and above the region in the \( xy \)-plane bounded by the curves \( y = x^{\frac{1}{2}} \), \( y = 0 \) and \( x = 1 \).

A simple sketch of the curve \( y = \sqrt{x} \) will show that our integral can be written as

\[
I = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz \, dy \, dx
\]

\[
= \int_0^1 \int_0^{\sqrt{x}} 6xy(1 + x + y) \, dy \, dx
\]

\[
= \int_0^1 \int_0^{\sqrt{x}} (6xy + 6x^2y + 6xy^2) \, dy \, dx
\]

\[
= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2}) \, dx
\]

\[
= \left( x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2} \right)_{x=0}^{x=1}
\]

\[
= 1 + \frac{3}{4} + \frac{4}{7}
\]

\[
= \frac{65}{28}
\]
PART II: Written (cont’)

15. (14pts) In the spherical coordinate system, find the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4 \cos \phi$.

The solid lies in

$$E = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 4 \cos \phi, 0 \leq \phi \leq \frac{\pi}{3}, 0 \leq \theta \leq 2\pi\}$$

Thus,

$$V = \iiint_E dv = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4 \cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} 64 \frac{3 \cos \phi \sin \phi}{3} d\phi d\theta$$

$$= 128\pi \left[ \cos^3 \phi \sin \phi \right]_0^{\pi/3}$$

$$= 128\pi \left[ 1 - \frac{1}{4} \right]$$

$$= 10\pi$$