King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 202 Exam I
Semester (121) October 2, 2012 Time: 8:30 - 10:30 pm

Name: ....................................................................
I.D: ....................................................... Section: ......

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1. (a) (3 points) Verify that $y = c_1 e^x + c_2 x e^x$ (c_1, c_2 constants) is a solution of $y'' - 2y' + y = 0$

Solution: $y' = c_1 e^x + c_2 e^x + c_2 x e^x$ 

$y'' = c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x$ 

$y'' - 2y' + y = c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x$ 

$-2c_1 e^x - 2c_2 e^x - 2c_2 x e^x$ 

$+ c_1 e^x + c_2 x e^x$ 

$= 0$ 

(b)(5-points) State the order of the ODE. Determine whether it is a linear or non-linear. Give a reason for your answer.

(i) $(1 - x)y' - 4xy = \sin x$ 

(ii) $x \frac{d^2y}{dx^2} - \left( \frac{dy}{dx} \right)^3 + y = 0$ 

(iii) $\frac{d^2u}{dr^2} + \frac{du}{dr} - u = \tan(r + 1)$ 

(iv) $\frac{d^2R}{dt^2} = -k/R^2$ 

(v) $x'' + \left( 1 + \frac{x'^2}{3} \right) (x') + x = 0$

(i) order = 1 / Linear 

(ii) order = 2 / Non-linear 

(iii) order = 2 / Linear 

(iv) order = 2 / Non-linear 

(v) order = 2 / Non-linear
2. (a) (2 points) Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$

Sln:  
\[ y' = \sec^2(x + c) \]
\[ 1 + y^2 = 1 + \tan^2(x + c) = \sec^2(x + c) \]
So  
\[ y' = 1 + y^2 \]  
(2 pt)

(b) (3 points) Find an explicit solution of the initial value problem
\[ y' = 1 + y^2, y(0) = 0 \]

Solution:  
By (a),  
\[ y = \tan(x + c) \]  
(1 pt)
\[ y(0) = 0 \implies \tan(c) = 0 \]  
(1 pt)
Can take $c = 0$
So  
\[ y = \tan x \]  
(1 pt)
(And so is  
\[ y = \tan(x + N\pi) \])  
(1 pt)

(c) (2 points) Determine the largest interval of definition of the solution you found in part (b).

Solution:  
For the solution in part (b)
\[ \tan x = \frac{\sin x}{\cos x} \], we want the largest interval $I$ containing the initial point $0$.
So  
\[ I = [-\pi/2, \pi/2] \]
3. (8 points) Solve the initial value problem

\[ y' + (\tan x)y = \cos^2 x, \quad y(0) = -1 \]

Note: You must work out all the integrals appearing in your solution.

\[
\text{Solution: Integrating factor is } \int \tan x \, dx = -\ln |\cos x|.
\]

\[
e^\int \tan x \, dx = e^{-\ln |\cos x|} = e^{-\ln |\cos x|} = e^{\ln |\cos x|^{-1}} = \frac{1}{|\cos x|}.
\]

Can take I.F as \( \frac{1}{\cos x} \)

Equation becomes:

\[
\frac{1}{\cos x} y' + \frac{\sin x}{\cos^2 x} = \cot x
\]

Or:

\[
\frac{d}{dx} \left( \frac{y}{\cos x} \right) = \cot x
\]

\[
\frac{y}{\cos x} = \int \cot x + C = \sin x + C
\]

\[
y = \cos x (\sin x + C)
\]

\[
y(0) = C = -1
\]

So, \( y = \csc x (\sin x - 1) \)

is the required solution \( \Box \)
4. (6 points) Find an integrating factor of the equation $6xy\,dx + (4y + 9x^2)\,dy = 0$ of the form $\mu = \mu(y)$.

**Solution:**

$6xy\,\mu(y)\,dx + \mu(y)(4y+9x^2)\,dy = 0$ is exact if

$$\frac{\partial}{\partial y} [6xy\,\mu(y)] = \frac{\partial}{\partial x} (\mu(y)(4y+9x^2))$$

So

$$6x\,(y\,\mu'(y) + \mu(y)) = \mu(y) \cdot 18x$$

$$y\,\mu'(y) + \mu(y) = 3\,\mu(y)$$

$$y\,\frac{d\mu}{dy} = 2\,\mu$$

$$\frac{d\mu}{\mu} = 2\,\frac{dy}{y}$$

$$\ln|\mu| = 2\,\ln|y| + C$$

$$|\mu| = e^{2\ln|y| + C}$$

$$|\mu| = e^{2\ln|y| + C} = y^2\,e^C$$

Can take $\mu = y^2$.
5. (10 points) Show that the following differential equation is exact and solve it.

\[(2y \sin x \cos x - y + 2y^2e^{xy^2})dx + (4xye^{xy^2} + \sin^2 x - x)dy = 0\]

**Solution:** Equation is exact if

\[
\frac{\partial}{\partial y} \left( 2y \sin x \cos x - y + 2y^2e^{xy^2} \right) = \frac{\partial}{\partial x} \left( 4xye^{xy^2} + \sin^2 x - x \right)
\]

1 pt

\[
\frac{\partial}{\partial y} \left( 2y \sin x \cos x - y + 2y^2e^{xy^2} \right) = 2 \sin x \cos x - 1 + 4y e^{xy^2} + 2y^2 e^{xy^2} \cdot 2xy \tag{x}
\]

2 pts

\[
\frac{\partial}{\partial x} \left( 4xye^{xy^2} + \sin^2 x - x \right) = 4y (e^{xy^2} + xe^{xy^2}) + 2 \sin x \cos x
\]

\[
= 4ye^{xy^2} + 4yxe^{xy^2} + 2 \sin x \cos x \tag{xx}
\]

So (x) = (xx) and the eqn is exact.

Want to find a function \( F \) with

\[
\frac{\partial F}{\partial x} = 2y \sin x \cos x - y + 2y^2e^{xy^2} \tag{xxxx}
\]

\[
\frac{\partial F}{\partial y} = 4xy e^{xy^2} + \sin^2(x) - x \tag{xxxx}
\]
Integrating the first equation we have

\[ F(x, y) = \int [2y \sin(x) + y + 2y^2 e^{xy^2}] \, dx \]

\[ = y \sin^2 x - yx + 2e^{xy^2} + h(y) \]

So

\[ \frac{\partial F}{\partial y} = \sin^2 x - x + 2e^{xy^2} (2xy) + h'(y) \]

Putting this equal to \( (\times) \) we get

\[ h'(y) = 0 \]

We can take \( h(y) = 0 \)

So

\[ F(x, y) = y \sin^2 x - yx + 2e^{xy^2} \]

and the solution of the given equation

\[ y \sin^2 x - yx + 2e^{xy^2} = C \]
6. (10 points) Find an explicit solution by separating variables of the initial value problem

\[
(\sqrt{x} + x) \frac{dy}{dx} = \sqrt{y} + y
\]

\[y(0) = 1.\]

Solution: \[
\frac{dy}{\sqrt{y} + y} = \frac{dx}{\sqrt{x} + x}
\]

\[2 \text{ pts} \left[ \int \frac{dy}{\sqrt{y} + y} = \int \frac{dx}{\sqrt{x} + x} \right] \quad (\star)
\]

To integrate, put \(\sqrt{x} = u\)

\[5 \text{ pts} \left\{ \begin{array} {l}
\text{so } x = u^2 \\
\int \frac{dx}{\sqrt{x} + x} = \int \frac{2u \, du}{u + u^2}
\end{array} \right. \]

\[= 2 \int \frac{du}{1+u} = 2 \ln (1+u)
\]

\[= 2 \ln (1+\sqrt{x})
\]

So (\(\star\)) gives

\[2 \ln (1+\sqrt{y}) = 2 \ln (1+\sqrt{x}) + C
\]

\[3 \text{ pts} \left\{ \begin{array} {l}
1 + \sqrt{y} = k (1 + \sqrt{x}) \\
y(0) = 1 \Rightarrow k = 2
\end{array} \right. \]

\[\text{So } 1 + \sqrt{y} = 2(1 + \sqrt{x})
\]
7. (6 points) Use suitable substitutions to change the following equations to linear differential equations.

DO NOT SOLVE THE NEW EQUATIONS.

(a) \( x^2 \frac{dy}{dx} + y^2 = xy \)

1 pt  Eqn. can be written as \( x^2 \frac{dy}{dx} - xy = -y^2 \), Bernoulli Type

1 pt  \( x^2 y^{-2} \frac{dy}{dx} - x y^{-1} = -1 \) (xy)

1 pt  Put \( u = y^{-1} \) \( \frac{du}{dx} = -y^{-2} \frac{dy}{dx} \)

So \( x/y \) becomes \( x^2 (\frac{du}{dx}) - xu = -1 \)

1 pt  or \( \frac{x^2}{u} \frac{du}{dx} + xu = 1 \)

(b) \( \frac{dy}{dx} - 1 - e^{y-2x+5} = 0 \)

1 pt  Put \( u = y - x + 5 \)

\( \frac{du}{dx} = \frac{dy}{dx} - 1 \)

So equation becomes

\( \frac{du}{dx} - e^u = 0 \)

1 pt  \( \frac{du}{dx} = e^u \) \( \frac{dx}{du} = e^{-u} \)

1 pt  which is linear in u.