King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Final Exam
The First Semester of 2012-2013 (121)

Time Allowed: 180 Minutes

Name: ______________________ ID#: ______________

Instructor: ______________________ Sec #: ________ Serial #: ________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		18
2		20
3		22
4		22
5		22
6		18
7		18
Total		140

Q:1 (18 points) Solve the initial vale problem using Laplace transform,

$y'' - 2y' + 2y = \delta(t-1)$ with $y(0) = 1,\ y'(0) = 0.$

Let $F(s) = \mathcal{L}\{y(t)\}$

We have $\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{\delta(t-1)\}$

$$s^2 F(s) - s\,y(0) - y'(0) - 2[sF(s) - y(0)] + 2F(s) = e^{-s} \quad 4\text{pts}$$

$$(s^2 - 2s + 2)F(s) - s + 2 = e^{-s}$$

$$F(s) = \frac{s-2}{s^2-2s+2} + \frac{e^{-s}}{s^2-2s+2} \quad 2\text{pts}$$

But, $s^2 - 2s + 2 = (s-1)^2 + 1$ 1pt

Thus, $$F(s) = \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1} + \frac{e^{-s}}{(s-1)^2+1} \quad 2\text{pts}$$

So, $$y(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s-1)^2+1}\right\}$$

$$\boxed{y(t) = \underbrace{e^{t}\cos t}_{3\text{pts}} - \underbrace{e^{t}\sin t}_{3\text{pts}} + \underbrace{e^{t-1}\sin(t-1)\,\mathcal{U}(t-1)}_{3\text{pts}}}$$

Q:2(20 points) Consider the Sturm–Liouville problem

$$y'' - 2y' + \lambda y = 0 \text{ with } y(0) = 0,\ y(2) = 0.$$

(a) Find the eigenvalues and corresponding eigenfunctions.

(b) Write the equation in self–adjoint form and write the weight function.

a) The auxiliary equation is $m^2 - 2m + \lambda = 0 \Rightarrow \Delta = 4 - 4\lambda$

Case 1: $\Delta = 0$, that is, $\lambda = 1$

$\Rightarrow m = 1$ and $y(x) = c_1 e^x + c_2 x e^x$

$y(0) = 0 \Rightarrow c_1 = 0$
$y(2) = 0 \Rightarrow c_2 = 0$ $\Big\} \Rightarrow y = 0$ (4pts)

Case 2: $\Delta > 0$, that is, $\lambda < 1$

$\Rightarrow m = 1 \pm \sqrt{1-\lambda}$ and $y(x) = c_1 e^{(1-\sqrt{1-\lambda})x} + c_2 e^{(1+\sqrt{1-\lambda})x}$ (4pts)

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Longrightarrow c_2 = -c_1$

$y(2) = 0 \Rightarrow c_1 e^{2(1-\sqrt{1-\lambda})} + c_2 e^{2(1+\sqrt{1-\lambda})} = 0$

Then $c_1\left(e^{2(1-\sqrt{1-\lambda})} - e^{2(1+\sqrt{1-\lambda})}\right) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$
$\Rightarrow y = 0$

Case 3: $\Delta < 0$, that is, $\lambda > 1$

$\Rightarrow m = 1 \pm i\sqrt{\lambda - 1}$ and $y(x) = \left(c_1 \cos\sqrt{\lambda-1}\,x + c_2 \sin\sqrt{\lambda-1}\,x\right)e^x$ (4pts)

$y(0) = 0 \Rightarrow c_1 = 0$

$y(2) = 0 \Rightarrow e^2 c_2 \sin(2\sqrt{\lambda-1}) = 0 \Rightarrow \sin 2\sqrt{\lambda-1} = 0$ and $c_2 \neq 0$

We find $2\sqrt{\lambda-1} = n\pi$ and $\boxed{\lambda_n = 1 + \frac{n^2\pi^2}{4}}$ (2pts) $n = 1, 2, 3 \ldots$

The corresponding eigenfunctions are $\boxed{y_n(x) = \sin\frac{n\pi}{2}x\, e^x}$ (2pts)

b)

$y'' - 2y' + \lambda y = 0$

$e^{-2x}y'' - 2e^{-2x} + \lambda e^{-2x} y = 0$

$\boxed{\frac{d}{dx}\left(e^{-2x}y'\right) + \lambda e^{-2x} y = 0}$ is the auto-adjoin form (3pts)

and $\boxed{p(x) = e^{-2x}}$ is the weight function. (1pt)

Q:3 (22 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,\ 0<r<1,\ 0<z<2,$$

subject to the boundary conditions

$$\begin{aligned} u(1,z) &= 0,\ 0<z<2 \\ u(r,0) &= 0,\ 0<r<1 \\ u(r,2) &= 1,\ 0<r<1 \\ \text{solution } u(r,z) \text{ is bounded at } r &= 0. \end{aligned}$$

Let $u(r,z)=R(r)\,Z(z)$. We find $R''Z+\frac{1}{r}R'Z+RZ''=0$ (2pts)

$\Leftrightarrow \frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$ (2pts) $\Leftrightarrow R''+\frac{1}{r}R'+\lambda R=0$, $Z''-\lambda Z=0$ (2pts)

$u(1,z)=0 \Rightarrow R(1)=0$ (1pt). Thus, $\begin{cases} rR''+R'+\lambda rR=0 \\ R(1)=0 \\ Z''-\lambda Z=0 \\ Z(0)=0 \end{cases}$

$u(r,0)=0 \Rightarrow Z(0)=0$ (1pt)

Let $\lambda=\alpha^2$. Then $rR''+R'+\alpha^2 rR=0 \Rightarrow R(r)=c\,J_0(\alpha r)$ (2pts)

And $R(1)=0 \Rightarrow J_0(\alpha_n)=0,\ n=1,2,3.$

On the other hand, $Z''-\alpha^2 Z=0 \Rightarrow Z(z)=c_1\cosh\alpha z+c_2\sinh\alpha z$

And $Z(0)=0 \Rightarrow c_1=0$ and $Z(z)=c\sinh\alpha z$ (2pts)

Thus, $\boxed{u(r,z)=\sum_{n=1}^{\infty}A_n\sinh\alpha_n z\,J_0(\alpha_n r)}$ (2pts)

$u(r,2)=1 \Rightarrow 1=\sum_{n=1}^{\infty}A_n\sinh 2\alpha_n\,J_0(\alpha_n r)$ (2pts)

So, $A_n\sinh 2\alpha_n=\frac{2}{J_1^2(\alpha_n)}\int_0^1 rJ_0(\alpha_n r)\,dr$ (2pts) $=\frac{2}{\alpha_n^2 J_1^2(\alpha_n)}\int_0^{\alpha_n}uJ_0(u)\,du$

But, $\frac{d}{du}[uJ_1(u)]=uJ_0(u) \Rightarrow \int_0^{\alpha_n}uJ_0(u)\,du=\alpha_n J_1(\alpha_n)$ (2pts)

Thus,

$$A_n\sinh 2\alpha_n=\frac{2}{J_1^2(\alpha_n)\alpha_n^2}\alpha_n J_1(\alpha_n)=\frac{2}{\alpha_n J_1(\alpha_n)}$$

$$\Rightarrow \boxed{A_n=\frac{2}{\alpha_n\sinh 2\alpha_n\,J_1(\alpha_n)}}$$ (2pts)