MATH 301.3 (Term 121)
Quiz 4 (Sects. 4.4-4.5) **Duration: 20mn**

Name: **ID number:**

1.)(4pts)Solve the IVP $y'' - 2y' = 1 + \delta(t-3)$, $y(0) = 1$, $y'(0) = -1$
2.)(3pts)Find $L^{-1}\{\frac{1}{(s^2+9)^2}\}$.
3.)(3pts)Use Laplace transform to solve $y'(t) = 2 - t^2 - \int_0^t y(\tau)d\tau$, $y(0) = 1$.

1.) $\mathcal{L}\{y''-2y'\} = \mathcal{L}\{1+\delta(t-3)\}$

$s^2F(s) - sy(0) - y'(0) - 2[sF(s) - y(0)] = \frac{1}{s} + e^{-3s}$

$(s^2-2s)F(s) = s-3 + \frac{1}{s} + e^{-3s}$

$F(s) = \frac{s^2-3s+1}{s^2(s-2)} + \frac{e^{-3s}}{s(s-2)}$

$= \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-2} + \left[\frac{d}{s} + \frac{e}{s-2}\right]e^{-3s}$

It is easy to find

$b = -\frac{1}{2}$, $c = -\frac{1}{4}$, $1 = a + c$

$\Rightarrow a = 5/4$

On the other hand, $d = -\frac{1}{2}$, $e = \frac{1}{2}$

So, $F(s) = \frac{5/4}{s} - \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{4}}{s-2} + \left[\frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2}\right]e^{-3s}$

$$\boxed{y(t) = \frac{5}{4} - \frac{1}{2}t - \frac{1}{4}e^{2t} + \left[-\frac{1}{2} + \frac{1}{2}e^{2(t-3)}\right]\mathcal{U}(t-3)}$$

2.) $L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \frac{1}{9}\sin 3t * \sin 3t$

$\sin 3t * \sin 3t = \int_0^t \sin 3\tau \sin(3t-3\tau)d\tau$

$= \frac{1}{2}\int_0^t [\cos(6\tau-3t) - \cos 3t]d\tau$

$= \frac{1}{2}\left[\frac{\sin(6\tau-3t)}{6} - \tau\cos 3t\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin 3t}{6} - t\cos 3t + \frac{\sin 3t}{6}\right]$

$= \frac{1}{2}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$

$\Rightarrow L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$

3.) $\mathcal{L}\{y'\} = \mathcal{L}\{2-t^2\} - \mathcal{L}\left\{\int_0^t y(\tau)d\tau\right\}$

$sF(s) - y(0) = \frac{2}{s} - \frac{2}{s^3} - \frac{F(s)}{s}$

$(s+\frac{1}{s})F(s) = 1 + \frac{2}{s} - \frac{2}{s^3}$

$F(s) = \frac{s}{s^2+1}\left(1 + \frac{2}{s} - \frac{2}{s^3}\right)$

$= \frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{2}{s^2(s^2+1)}$

$= \frac{s}{s^2+1} + \frac{2}{s^2+1} - \left(\frac{2}{s^2} - \frac{2}{s^2+1}\right)$

$= -\frac{2}{s^2} + \frac{s}{s^2+1} + \frac{4}{s^2+1}$

$$\Rightarrow \boxed{y(t) = -2t + \cos t + 4\sin t}$$

MATH 301.2 (Term 121)

Quiz 4 (Sects. 4.4-4.5) Duration: 20mn

Name: **ID number:**

1.)(4pts)Solve the IVP $y'' + 4y' + 5y = \delta(t - 2\pi)$, $y(0) = -1$, $y'(0) = 1$

2.)(3pts)Find $L^{-1}\{\frac{1}{s(s+1)^4}\}$.

3.)(3pts)Use Laplace transform to solve $f(t) = e^{-t} + t^2 - \int_0^t f(\tau)d\tau$.

1.) $\mathcal{L}\{y''+4y'+5y\} = \mathcal{L}\{\delta(t-2\pi)\}$

$s^2F(s) - sy(0) - y'(0) + 4[sF(s) - y(0)] + 5F(s) = e^{-2\pi s}$

$(s^2+4s+5)F(s) = -s-3 + e^{-2\pi s}$

$$F(s) = \frac{-s-3}{s^2+4s+5} + \frac{e^{-2\pi s}}{s^2+4s+5}$$

$$= \frac{-(s+2)-1}{(s+2)^2+1} + \frac{e^{-2\pi s}}{(s+2)^2+1}$$

$\Rightarrow y(t) = -e^{-2t}\cos t - e^{-2t}\sin t + \left(e^{-2(t-2\pi)}\sin(t-2\pi)\right)\mathcal{U}(t-2\pi)$

$$\boxed{y(t) = -e^{-2t}(\cos t + \sin t) + e^{-2(t-2\pi)}\sin t\,\mathcal{U}(t-2\pi)}$$

2.) $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^4}\right\} = \int_0^t f(\tau)\,d\tau$, where

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}\right\}$

$= \frac{1}{6}t^3e^{-t}$

So $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^4}\right\} = \frac{1}{6}\int_0^t \tau^3 e^{-\tau}\,d\tau$

Integration by part 3 times

$$\boxed{= -\frac{1}{6}(t^3+3t^2+6t+6)e^{-t} + 1}$$

3.) $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}+t^2\} - \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$

$$F(s) = \frac{1}{s+1} + \frac{2}{s^3} - \frac{F(s)}{s}$$

$$\left(1+\frac{1}{s}\right)F(s) = \frac{1}{s+1} + \frac{2}{s^3}$$

$$F(s) = \frac{s}{s+1}\left(\frac{1}{s+1} + \frac{2}{s^3}\right)$$

$$= \frac{s}{(s+1)^2} + \frac{2}{s^2(s+1)}$$

$$= \frac{1}{s+1} - \frac{1}{(s+1)^2} + 2\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right]$$

$$= -\frac{2}{s} + \frac{2}{s^2} + \frac{3}{s+1} - \frac{1}{(s+1)^2}$$

$$\Rightarrow \boxed{f(t) = -2 + 2t + 3e^{-t} - e^{-t}t}$$