MATH 301.2 (Term 121)

Quiz 6 (Sects. 13.3-13.5) Duration: 20mn

Name: ID number:

Use separation of variable to solve the following boundary value problem

$$\begin{cases} k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, & 0 < x < L, \quad t > 0, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0, & t > 0, \\ u(x,0) = f(x).. \end{cases}$$

$u(x,t) = X(x)T(t)$

$kX''T = XT'$

$\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda$

$X'' + \lambda X = 0$

$T' + k\lambda T = 0$

$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow X'(0) = 0$

$\frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow X'(L) = 0$

case 1 : $\boxed{\lambda = 0}$

$X'' = 0 \Rightarrow X(x) = C_1 + C_2 x$

$X'(x) = C_2$, $X'(0) = 0 \Rightarrow C_2 = 0$

$X'(L) = 0 \Rightarrow C_2 = 0$

$\Rightarrow X(x) = C_1$

$T' = 0 \Rightarrow T(t) = C$

$\Rightarrow u(x,t) = A_0$

Case 2 : $\lambda < 0$, $\boxed{\lambda = -\alpha^2}$

$X'' - \alpha^2 X = 0 \Rightarrow X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$

$X'(x) = C_1 \alpha \sinh \alpha x + C_2 \alpha \cosh \alpha x$

$X'(0) = 0 \Rightarrow \alpha C_2 = 0 \Rightarrow C_2 = 0$

$X'(L) = 0 \Rightarrow \alpha C_1 \sinh \alpha L = 0 \Rightarrow C_1 = 0$

$\Rightarrow X(x) = 0$ and $u(x,t) = 0$

Case 3: $\lambda > 0$, $\boxed{\lambda = \alpha^2}$

$X'' + \alpha^2 X = 0 \Rightarrow X = C_1 \cos \alpha x + C_2 \sin \alpha x$

$X'(x) = -C_1 \alpha \sin \alpha x + C_2 \alpha \cos \alpha x$

$X'(0) = 0 \Rightarrow C_2 = 0$

$X'(L) = 0 \Rightarrow \sin \alpha L = 0$, $\alpha L = n\pi$

$\Rightarrow X(x) = C \cos \frac{n\pi}{L} x$

$T' + k\alpha^2 T = 0 \Rightarrow T(t) = C e^{-k\alpha^2 t} = C e^{-k\frac{n^2\pi^2}{L^2} t}$

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x \, e^{-k\frac{n^2\pi^2}{L^2} t}}$$

when $t = 0$

$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x$

$\Rightarrow A_0 = \dfrac{\int_0^L f(x)\,dx}{\int_0^L 1^2\,dx} = \dfrac{1}{L}\int_0^L f(x)\,dx$

$A_n = \dfrac{\int_0^L f(x) \cos \frac{n\pi}{L} x\,dx}{\int_0^L \cos^2 \frac{n\pi}{L} x\,dx} = \dfrac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x\,dx$

MATH 301.3 (Term 121)

Quiz 6 (Sects. 13.3-13.5) Duration: 20mn

Name: **ID number:**

Use separation of variable to solve the following boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \pi, \quad 0 < y < \pi, \\ \dfrac{\partial u}{\partial x}(0, y) = u(0, y), \;\; u(\pi, y) = 1, \\ u(x, 0) = 0, \;\; u(x, \pi) = 0. \end{cases}$$

$u(x,y) = X(x)Y(y)$

$X''Y + XY'' = 0$

$\dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda$

$X'' + \lambda X = 0$

$Y'' - \lambda Y = 0$

$u(x,0) = 0 \Rightarrow Y(0) = 0$

$u(x,\pi) = 0 \Rightarrow Y(\pi) = 0$

Case1: $\boxed{\lambda = 0}$

$Y'' = 0 \Rightarrow Y(y) = c_1 + c_2 y$

$Y(0) = 0 \Rightarrow c_1 = 0$

$Y(\pi) = 0 \Rightarrow c_2 = 0$

$\Rightarrow Y = 0$ and $u(x,y) = 0$

Case2: $\lambda > 0$, $\boxed{\lambda = \alpha^2}$

$Y'' - \alpha^2 Y = 0 \Rightarrow Y(y) = c_1 \cosh \alpha y + c_2 \sinh \alpha y$

$Y(0) = 0 \Rightarrow c_1 = 0$

$Y(\pi) = 0 \Rightarrow c_2 \sinh \alpha\pi = 0 \Rightarrow c_2 = 0$

$\Rightarrow Y(y) = 0$ and $u(x,y) = 0$

Case3: $\lambda < 0$, $\lambda = -\alpha^2$

$Y'' + \alpha^2 Y = 0 \Rightarrow Y(y) = c_1 \cos \alpha y + c_2 \sin \alpha y$

$Y(0) = 0 \Rightarrow c_1 = 0$

$Y(\pi) = 0 \Rightarrow c_2 \sin \alpha\pi = 0 \Rightarrow \alpha\pi = n\pi$, $\alpha = n$, $n = 1, 2, \ldots$

$\Rightarrow Y(y) = c_2 \sin ny$

$X'' - \alpha^2 X = 0 \Rightarrow X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$
$= c_1 \cosh nx + c_2 \sinh nx$

$\Rightarrow \boxed{u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh nx + B_n \sinh nx) \sin ny}$

$\dfrac{\partial u}{\partial x}(x,y) = \sum_{n=1}^{\infty} (n A_n \sinh nx + n B_n \cosh nx) \sin ny$

$\dfrac{\partial u}{\partial x}(0,y) = u(0,y) \Rightarrow$

$\sum_{n=1}^{\infty} n B_n \sin ny = \sum_{n=1}^{\infty} A_n \sin ny$

$\Rightarrow \boxed{n B_n = A_n}$

$u(\pi, y) = 1 \Rightarrow 1 = \sum_{n=1}^{\infty} (A_n \cosh n\pi + B_n \sinh n\pi) \sin ny$

$A_n \cosh n\pi + B_n \sinh n\pi = \dfrac{\int_0^\pi \sin ny \, dy}{\int_0^\pi \sin^2 ny \, dy} = -\dfrac{2}{\pi n}\left((-1)^n - 1\right)$

$\boxed{B_n = \dfrac{-1}{(n \cosh n\pi + \sinh n\pi)} \cdot \dfrac{2}{\pi n}\left((-1)^n - 1\right)}$