(1) Consider the graph $G$ in the figure: (20pts)

(a) Is the graph $G$ bipartite? If yes, give a bipartition. If not, explain why not?
(b) How many bridges does $G$ have? List them.
(c) How many cut-vertices does $G$ have? List them, then find their degrees in $\overline{G}$.
(d) List all values of $n$ for which $K_n$ is a subgraph of $G$?
(e) Find a subgraph that is not an induced subgraph of $G$?
(f) How many blocks does $G$ have? Draw them.
(g) Find a spanning subgraph having a minimum number of edges?
(2) Determine if the following statements are TRUE or FALSE. If a statement is true, sketch the proof; if it is false, give a counterexample. (25pts)

(a) If $G$ and $H$ are two simple graphs with same degree-sequence, then they are isomorphic.

(b) If $G$ is bipartite, then $\overline{G}$ is bipartite.

(c) If $G$ is connected, then $\overline{G}$ is disconnected.

(d) If $G$ is regular, then $\overline{G}$ is regular.

(e) If $v$ is a cut-vertex of a simple graph $G$, then $v$ is a cut-vertex of $\overline{G}$

(f) If $G$ is a nontrivial connected graph such that each vertex is of even degree, then $G$ has at least one circuit.

(g) The following two graphs are isomorphic.
(3) Find each of the following: (30pts)

(a) The maximum number of bridges in a tree $T$ with $n$ vertices.

(b) The maximum number of vertices in a graph $G$ with 15 edges and 3 components.

(c) The minimum number of cut vertices in a tree $T$ with $n$ vertices.

(d) All trees $T$ whose complement $\overline{T}$ is also a tree.

(e) The number of labeled spanning forests in $\overline{K_{4,4}}$.

(f) The labeled tree having Pr"ufer sequence $(4, 5, 7, 2, 1, 1, 6, 6, 7)$.

(g) $x$ and $y$ (by using Havel-Hakimi Theorem) if $S : 7, 5, 3, x, 2, 2, 2, y, 1, 1, 1$ is a graphical degree sequence.
(4) Answer only 2 problems

(a) Prove that a simple graph $G$ with $2n$ vertices such that the degree of each vertex is at least $n$ is connected.

(b) Show that a tree $T$ has one and only one vertex in the center if and only if $diam(T) = 2 \cdot rad(T)$.

(c) Prove Cayley’s Tree Formula (for each positive integer $n$, there are $n^{n-2}$ distinct labeled trees of order $n$ having the same vertex set) as a corollary to the Matrix-Tree Theorem.

(d) Prove that every nontrivial connected graph contains at least two vertices that are not cut-vertices.