Q1. Let \( P \) be the matrix \( \frac{aa^T}{a^Ta} \) where \( a \) is a vector.
   a) Show that \( P^2 = P \).
   b) Is \( P \) invertible? Why?
   c) Show that the trace of \( P \) always equals one.
Q2. Let W be a subspace of $\mathbb{R}^3$ spanned by the set $S = \{x_1, x_2, x_3\}$, where

\[ x_1^T = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \end{bmatrix}, \quad x_2^T = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{2} \end{bmatrix}, \quad x_3^T = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \end{bmatrix} \]

a) Use Gram-Schmidt to find an orthonormal basis for W.

b) Find the QR-factorization of the matrix $A = [x_1 \quad x_2 \quad x_3]$.

c) Use part (b) to solve the system $Ax=b$ where $b^T = [1 \quad 0 \quad -1]$.
Q3. A small company has been in business for three years and has recorded annual profits (in thousands of dollars) as follows

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>7</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Assuming that there is a linear trend in the declining profits, predict the year and the month in which the company begins to lose money.

Q4. Find the next Legendre Polynomial – a cubic orthogonal to 1, x, and $x^2 - \frac{1}{3}$ over the interval [-1, 1].
Q5. Find the pivots (without using the elimination) of the matrix

\[
A = \begin{bmatrix}
2 & 1 & 2 \\
4 & 5 & 0 \\
2 & 7 & 0 \\
\end{bmatrix}.
\]

Q6. If \( A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \) find \( A^{100} \) by diagonalizing \( A \).
Q7. (a) Let $A$ be a symmetric matrix. Show that all eigenvalues of $A$ are real.
(b) Let $v \in \mathbb{C}^n$. Show that if $v^*v = 1$, then $I - 2vv^*$ is hermitian and unitary.

Q8. Suppose $T$ is a $3 \times 3$ upper triangular matrix with entries $t_{ij}$. Compare the entries of $TT^H$ and $T^HT$, and show that if they are equal then $T$ must be diagonal.
Q9. Find a unitary U and a triangular T so that \( U^{-1}AU = T \), for \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \).
Q10. Find the Jordan form $J$ and the matrix $M$ for $A=[0 \ 0 \ 0; 1 \ 0 \ 0; 2 \ 1 \ 0]$. What is the solution to $\frac{du}{dt} = Au$ and what is $e^{At}$?
Q11. Label each of the following statements as True or False.
(a) If the vectors x and y are orthogonal and P is a projection matrix, then Px and Py are also orthogonal.
(b) If all eigenvalues of a real matrix are equal to 1 then the matrix must be orthogonal.
(c) The sum of two unitary matrices must be unitary.
(d) If N is a unitary matrix, then there exists a unitary matrix U such that $U^{-1}NU$ is a diagonal matrix.
(e) If N is normal, then $\|Nx\| = \|N^Hx\|$ for every vector x.