King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
Final Exam – Semester I, 2012-2013

Problem 1. Expand

$$f(z) = \frac{z^2+9z+11}{(z+1)(z+4)}$$

as a Laurent series about $z=0$ when

$$(a)\ |z|<1 \qquad (b)\ 1<|z|<4$$

Problem 2. For $\phi \in (0,\pi)$ and $n \in \mathbb{N}$, show that

$$\frac{1}{2\pi i}\oint_{|z|=2} \frac{z^n}{1-2z\cos\phi+z^2}\,dz = \frac{\sin n\phi}{\sin\phi}.$$

Problem 3. Compute the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx.$

Problem 4. (Generalized Argument Principle)

(1) Let f be meromorphic on a neighborhood of the closed unit disk $\overline{\mathbb{D}}(0,1)$, and that f has neither poles nor zeros on $\partial\mathbb{D}(0,1)$. Let g analytic on a neighborhood of the closed unit disk $\overline{\mathbb{D}}(0,1)$.
Prove that

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,1)} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{p} g(z_j)n_j - \sum_{k=1}^{q} g(w_k)m_k,$$

where $n_1, n_2, \ldots, n_p$ are the multiplicities of the zeros $z_1, z_2, \ldots, z_p$ of f in $\mathbb{D}(0,1)$ and $m_1, m_2, \ldots, m_q$ are the multiplicities of the poles $w_1, w_2, \ldots, w_q$ of f in $\mathbb{D}(0,1)$

(2) *Application:* Suppose f is analytic in $\mathbb{D}(0,2)$ and

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,1)} \frac{f'(z)}{f(z)}\,dz = 2,$$

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,1)} z\,\frac{f'(z)}{f(z)}\,dz = i,$$

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}(0,1)} z^2\,\frac{f'(z)}{f(z)}\,dz = -\frac{1}{2}.$$

Find the zeros of f in the unit disk $\mathbb{D}$.

Problem 5. Show that the polynomial $z^5 + 15z + 1$ has all its roots in the disk $\mathbb{D}(0, 2)$ but only one of these roots lies in the disk $\mathbb{D}(0, 3/2)$.

Problem 6.

(1) Find all entire functions f such that $|f(z)| \leq e^x$ for all $z = x + iy \in \mathbb{C}$.
(2) Find all entire functions f such that $|f(z)| \leq |\sin z|$ for all $z \in \mathbb{C}$, how about if $\sin z$ is replaced by $\cos z$?

Problem 7. Let f be analytic in $|z| < 1$ with $f(0) = 0$ and $|f(z)| < 1$.
Prove that

$$F(z) = f(z) + f(z^2) + f(z^3) + \ldots = \sum_{n \geq 1} f(z^n)$$

is analytic in $|z| < 1$, and that

$$|F(z)| \leq \frac{r}{1-r}, \quad |z| \leq r < 1.$$

Bonus Problem (Parseval-Gutzmer formula)

Let f be an analytic function on $\overline{\mathbb{D}}(0, r)$, the closed disk of radius r, with Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Prove that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = 2\pi \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

Deduce that

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \leq M_r{}^2,$$

where $M_r = \sup\{|f(z)| : |z| = r\}$.